UNIT - II

ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER

* INTRODUCTION:-

let the $y = f(x)$ is a function. then the derivatives $\frac{dy}{dx}, \frac{d^2y}{dx^2}, \frac{d^3y}{dx^3}$ ------ are measures of the rate at which y changes, rate at which $\frac{dy}{dx}$ changes, rate at which $\frac{d^2y}{dx^2}$ changes ---- with respect to x.

→ This y may be a certain distance and x may time.

→ y may be a certain ~~wait~~ ~~sem~~ area changing w.r.t some x.

→ y may be price of a commodity changing w.r.t price x of some other commodity.

→ y may be temperature changing w.r.t time x.

So, there can be some equation relating $x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2}$ ---- in the form.

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} ---\right) = 0.$$

with given conditions on y and the derivatives where from the unknown function y is to be determined.

This is a typical problem under O.D.Eq's.

where x - independent variable
y - dependent variable.

* DIFFERENTIAL EQUATION :-

An equation involving differentials is called a differential equation.

(OR) An equation involving derivatives of one dependent variable with respect to one or more independent variables is called a differential equation.

Differential Equations are two types.

there are

(1) Ordinary differential equations

(2) Partial differential equations

* ORDINARY DIFFERENTIAL EQUATION :-

A differential equation is said to be ordinary, if the derivatives in that equation have reference to a single independent variable.

Ex:- (i) $\frac{d^2y}{dx^2} + \frac{dy}{dx} + 7y = \cos x$

(ii) $(x^2+y^2)\,dy + y\,dx = 0$

(iii) $\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^2 + y = 1.$ etc..

The general form of an ordinary diff. eqn is

$$f\left(x, y, \frac{dy}{dx}, \frac{d^2y}{dx^2} \cdots \frac{d^ny}{dx^n}\right) = 0.$$

* $f\left(x, y, \frac{dy}{dx}\right) = 0$ (or) $f(x, y, y') = 0$ is 1^{st} order Dif. Eqn.

* $f(x, y, y', y'') = 0$ is 2^{nd} order Diffn. eqn.

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* PARTIAL DIFFERENTIAL EQUATIONS :-

A differential equation is said to partial, if the derivatives in that equation have reference to two or more independent variables.

Ex:- $\frac{\partial^2 y}{\partial x^2} = \frac{1}{c^2}\frac{\partial^2 y}{\partial t^2}$ (1-D wave eqn)

$\frac{\partial^2 u}{\partial x^2} = \frac{1}{c^2}\frac{\partial u}{\partial t}$ (1-D heat conduction eqn)

$\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} = 0$ (2-D Laplace's eqn).

* ORDER OF A DIFFERENTIAL EQUATION :-

A differential equation is said to be of "order n", if n^{th} order derivative is the highest order derivative in that equation.

i.e, The order of the highest order derivative involved in a differential eqn. is called the order of the diffn. eqn.

Ex:- $x^2\frac{d^2y}{dx^2} + \frac{dy}{dx} + y = x$, order = 2.

$\frac{d^3y}{dx^3} + \left(\frac{dy}{dx}\right)^4 + y = 4\cos x$, order = 3.

* DEGREE OF A DIFFERENTIAL EQUATION :-

let $F(x, y, y', y'', \ldots, y^{(n)}) = 0$ be a diffn. eqn of order n which is free from radicals and fractions as far as the derivatives are concerned. If the given differential eq is a polynomial in $y^{(n)}$ then the highest degree of $y^{(n)}$ is defined as the degree of the diffn. eqn.

Ex:- (i) $y = x\frac{dy}{dx} + \sqrt{1+\left(\frac{dy}{dx}\right)^2} \Rightarrow (1-x^2)\left(\frac{dy}{dx}\right)^2 + 2xy\frac{dy}{dx} + (1-y^2)$

degree = 2.

(ii) $a\frac{d^2y}{dx^2} = \left[1+\left(\frac{dy}{dx}\right)^2\right]^{3/2} \Rightarrow a\left(\frac{d^2y}{dx}\right)^2 = \left[1+\left(\frac{dy}{dx}\right)^2\right]^3$

degree = 2.

* FORMA

* DEFINATIONS OF SOLUTIONS OF A DIFF^n. EQ^{n}S:-

* (i) SOLUTION :- Any relation b/w the dependent and independent variables not containg their derivatives, which satisfies the given diff^n. eq^n is called a solution (or) integral of the diff^n. eq^n.

Ex:- $y = A\cos x + B\sin x$ is soln of $\frac{d^2y}{dx^2} + y = 0$. [A, B are arbitary constants]

* (ii) General Solution :- A solution containing the number of independent arbitary constants which is equal to the order of the diff^n. eq^n is called the general solution (or) complete primitive of the equation.

Ex:- $y = c_1e^{x} + c_2e^{2x}$ is general soln of $y'' - 3y' + 2y = 0$ [c_1, c_2 are Indepent arbitary constants]

* (iii) PARTICULAR SOLUTION :- A solution obtained from the general soln of Diff^n eq^n by giving particular values to the independent arbitary constants is called particular solution to the given diff^n. eq^n.

Ex:- Some Particular solns of $y'' - 3y + 2y = 0$ are $y = e^{x} + e^{2x}$, $y = e^{x} - e^{2x}$ ...

* (iv) Singular solution:- A solution which cannot be obtained from any general solution of a Diff^n eq^n by any choice of the independent arbitary constants is called a singular solution of Diff^n eq^n.

Ex:- $y = (x+c)^2$ is gen. soln of $y_1^2 - 4y = 0$ and $y = 0$ is also singular solution

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* FORMATION OF DIFFERENTIAL EQUATION :-

In general an ordinary diffn eqn is obtained by eliminating the arbitary constants $c_1, c_2 --- c_n$ from a relation like $\phi(x, y, c_1, c_2 --- c_n) = 0$

Now let $\phi(x, y, c_1, c_2 --- c_n) = 0$ ——— ①

where $c_1, c_2 --- c_n$ are arbitary constants. Differentiating eq ① successively w.r.t x, n times, and eliminating the n arbitary constants $c_1, c_2 --- c_n$ from the above $(n+1)$ equations, we obtain the differential equation $F(x, y, y_1, y_2 --- y_n) = 0$. It's general solution is given by the relation ① itself.

PROBLEMS :-

(1) Form the diffn. eqn by eliminating arbitary constants $y = Ae^{x} + Be^{-x}$

Sol: Given $y = Ae^{x} + Be^{-x}$, where A, B are arbitary constants

Dn. w.r.t x

$$\frac{dy}{dx} = Ae^{x} - Be^{-x}$$

again Dn.

$$\frac{d^2y}{dx^2} = Ae^{x} + Be^{-x}$$

By eliminating A, B, $\frac{d^2y}{dx} = y$

$$\therefore \frac{d^2y}{dx^2} - y = 0.$$

which is a required Dn. Eqn.

(2) Form the differential equation by eliminating arbitary constants $y^2 = (x-c)^3$.

<u>Sol:</u>

Given $y^2 = (x-c)^3$, ——— ① c is arbitary constant

Diffⁿ w.r.t x

$$2yy' = 3(x-c)^2$$

Squaring, $$4y^2y'^2 = 9(x-c)^3(x-c)$$

$$\frac{4}{9}y'^2 = (x-c)$$

from ①

$$\therefore y^2 = \left(\frac{4}{9}y'^2\right)^3$$

$$= \frac{64}{729}y'^6$$

$$\therefore 64y_1^6 = 729y^2$$

<u>H.W.P:-</u>

1) Form the diffⁿ eqⁿ by eliminating arbitary constants $y = Ae^{-3x} + Be^{2x}$

$y_2 + y_1 - 6y = 0$

2) Form the diffⁿ eqⁿ by eliminating arbitary constants $y = ax^3 + bx^2$.

$+x^2y_2 - 4xy_1 + 6y = 0$

* 3) what is the D.Eqⁿ satisfied by $y = mx$ after eliminating m ?

* 4) Obtain the D.Eqⁿ of the family of straight lines having nonzero intercepts on the coordinate axes.

* ORDINARY DIFFERENTIAL EQUATIONS OF FIRST ORDER AND OF FIRST DEGREE :-

* DEFINATION :- An equation of the form $\frac{dy}{dx} = f(x,y)$ is called a ~~differenti~~ ordinary differential equation of the first order and of the first degree. where $f(x,y)$ is a function in x and y.

Now dissoussing about to find the general solution of $\frac{dy}{dx} = f(x,y)$ by using following methods:

(1) Variables separable.

(2) Homogeneous equations and equations reducible to homogeneous form.

(3) Exact equations and those which can be made exact by use of integrating factors.

(4) Linear equation and Bernoulli's equation.

*I VARIABLES SEPARABLE :-

If the differential equation $\frac{dy}{dx} = f(x,y)$ can be expressed in the form

$$\frac{dy}{dx} = \frac{f(x)}{g(y)} \quad (or) \quad f(x)\,dx = g(y)\,dy \qquad \text{—— (1)}$$

where f and g are continuous functions of a single variable, then it is said to be of the form "variables separable".

Integrating the equation (1), the general solution is $\int f(x)\,dx = \int g(y)\,dy + c$, where c is any arbitary constant.

PROBLEMS :-

1) Solve $e^{x-y}dx + e^{y-x}dy = 0$

Sol: Given $e^{x-y}dx + e^{y-x}dy = 0$

$$e^{x-y}dx = -e^{y-x}dy$$

$$\frac{e^x}{e^y}dx = -\frac{e^y}{e^x}dy$$

Variables Separable, $e^{2x}dx = -e^{2y}dy$

$$\int e^{2x}dx = -\int e^{2y}dy$$

$$\frac{e^{2x}}{2} = -\frac{e^{2y}}{2} + \frac{c}{2}$$

General Sol: is $e^{2x} + e^{2y} = c$.

2) Solve $\frac{dy}{dx} = (4x+y+1)^2$

Sol: Given $\frac{dy}{dx} = (4x+y+1)^2$ (∵ there is no separating)

put $4x+y+1 = z \Rightarrow 4 + \frac{dy}{dx} = \frac{dz}{dx} \Rightarrow \frac{dy}{dx} = \frac{dz}{dx} - 4$

$$\therefore \frac{dz}{dx} - 4 = z^2 \Rightarrow \frac{dz}{dx} = z^2 + 4$$

Variable separable, $\int \frac{1}{z^2+4}dz = \int dx$

$$\frac{1}{2}\tan^{-1}\frac{z}{2} = x + c$$

$$z = 2\tan(2x+2c$$

G. Soln: $4x+y+1 = 2(\tan(2x+2c$

H.W.P :-

(1) Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$ $\quad \sin x(e^y+1) = c$

(2) Solve $\frac{dy}{dx} = e^{2x-3y} + x^2 \cdot e^{-3y}$ $\quad 3e^{2x} + 2x^3 - 2e^{3y} = c$

(3) Solve $(x+y)^2\frac{dy}{dx} = a^2$ $\quad y = a\tan^{-1}\left(\frac{x+y}{a}\right) + c$

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II HOMOGENEOUS DIFFERENTIAL EQUATIONS :

* HOMOGENEOUS FUNCTION :-

A function $f(x,y)$ is said to be a "homogenous fn" in x and y of degree n, if $f(kx,ky)=k^n f(x,y)$ for all values of k, where n is a real number.

* Examples :-

(i) $f(x,y) = \dfrac{x^2+y^2}{x^3+y^3}$ is H.F of degree "-1".

(ii) $f(x,y) = \dfrac{x^2+y^2 e^{x/y}}{x+y}$ is H.F of degree 1.

(iii) $f(x,y) = \dfrac{x+y}{x-y}$ is H.F of degree '0'.

(iv) $f(x,y) = \cos x + \sin y$ is not H.F.

* NOTE :- A H.F of degree n in x and y can also be expressed as $y^n f\left(\frac{x}{y}\right)$ (or) $x^n f\left(\frac{y}{x}\right)$.

Ex :- $f(x,y) = ax^3 + 3bx^2y + cy^3 \Rightarrow f(x,y) = y^3\left[a\left(\frac{x}{y}\right)^3 + 3b\left(\frac{x}{y}\right)^2 + c\right]$

is H.F of degree 3.

* HOMOGENEOUS DIFFERENTIAL EQUATION :-

A differential equation $\frac{dy}{dx} = f(x,y)$ of first order and first degree is called homogeneous in x and y, if the function $f(x,y)$ is a H.F of degree '0' in x and y.

Ex :- (i) $f(x,y) = \dfrac{x+y}{x-y}$

(ii) $f(x,y) = \dfrac{x^3+y^3}{2x^2y}$

(iii) $f(x,y) = \dfrac{x^2-y^2}{xy}$

* GENERAL SOLUTION OF H.D.E :-

let H.D.Eq $\frac{dy}{dx} = f(x,y)$ — ①

we write $f(x,y) = \phi(\frac{y}{x})$ (or) $\frac{y}{x}$

$\Rightarrow \frac{dy}{dx} = \phi(\frac{y}{x})$ — ② [∵ f is H.F]

put $\frac{y}{x} = v$ (or) $y = vx$

$\frac{dy}{dx} = v + x\frac{dv}{dx}$ — ③

from eq ②,③, $v + x\frac{dv}{dx} = \phi(v)$

$x\frac{dv}{dx} = \phi(v) - v$

variables separable

$\frac{1}{\phi(v)-v}\,dv = \frac{1}{x}\,dx$

Integrating, $\int \frac{1}{\phi(v)-v}\,dv = \int \frac{1}{x}\,dx + c$

after getting solution, again put $v = \frac{y}{x}$.

* PROBLEMS :-

1) Solve $(x^2+y^2)\,dx = 2xy\,dy$

Sol: Given $\frac{dy}{dx} = \frac{x^2+y^2}{2xy}$ — ① is H.D.Eq.

put $y = vx$. $\Leftrightarrow$ $\frac{dy}{dx} = v + x\frac{dv}{dx}$ — ②

From ①,②, $v + x\frac{dv}{dx} = \frac{x^2+v^2x^2}{2x.vx}$

$v + x\frac{dv}{dx} = \frac{1+v^2}{2v} \Rightarrow x\frac{dv}{dx} = \frac{1+v^2-2v^2}{2v} = \frac{1-v^2}{2v}$

$\frac{2v}{1-v^2}\,dv = \frac{1}{x}\,dx$

$-\int \frac{-2v}{1-v^2}\,dv = \int \frac{1}{x}\,dx$

$-\log(1-v^2) = \log x + \log c$

$x(1-v^2) = c$

$x\left(1-\frac{y^2}{x^2}\right) = c$

$\therefore x^2 - y^2 = cx$

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(Q) solve $(x^2-4xy-2y^2)\,dx + (y^2-4xy-2x^2)\,dy = 0$.

Sol: Given $\frac{dy}{dx} = \frac{2y^2+4xy-x^2}{y^2-4xy-2x^2}$ is H.D.Eq ——(1)

put $\frac{y}{x} = v \Rightarrow y = vx$

$\therefore \frac{dy}{dx} = v + x\frac{dv}{dx}$ ——(2)

From eq (1), (2), $v + x\frac{dv}{dx} = \frac{2v^2x^2+4x.vx-x^2}{v^2x^2-4x.vx-2x^2}$

$$v + x\frac{dv}{dx} = \frac{2v^2+4v-1}{v^2-4v-2}$$

$$x\frac{dv}{dx} = \frac{2v^2+4v-1-v^3+4v^2+2v}{v^2-4v-2} = \frac{-v^3+6v^2+6v-1}{v^2-4v-2}$$

$$\therefore \frac{v^2-4v-2}{-v^3+6v^2+6v-1}\,dv = \frac{1}{x}\,dx$$

$$-\frac{1}{3}\int \frac{-3v^2+12v+6}{-v^3+6v^2+6v-1}\,dv = \int \frac{1}{x}\,dx$$

$$-\frac{1}{3}\log(-v^3+6v^2+6v-1) = \log x - \log c$$

$$(-v^3+6v^2+6v-1)^{1/3}\cdot x = c$$

$$\left(-\frac{y^3}{x^3}+6\frac{y^2}{x^2}+6\frac{y}{x}-1\right)x^3 = c^3$$

$$\therefore -y^3+6xy^2+6x^2y-x^3 = c^3$$

* H.W.P:-

1) Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$. $\quad -\frac{1}{3}\frac{x^3}{y^3} + \log y = \log c$

2) Solve $(y^2-2xy)\,dx = (x^2-2xy)\,dy$ $\quad xy^2 - x^2y = c$

* NON-HOMOGENEOUS D. Eq:-

If a_1, b_1, c_1 and a_2, b_2, c_2 are constants atleast one of c_1, c_2 is not zero then $(a_2x+b_2y+c_2)\frac{dy}{dx} = (a_1x+b_1y+c_1)$ (or) $\frac{dy}{dx} = \frac{a_1x+b_1y+c_1}{a_2x+b_2y+c_2}$ is called non-homogeneous differential equation of the first order and the first degree in x & y.

(i) $\frac{a_1}{a_2} \neq \frac{b_1}{b_2}$ (ii) $\frac{a_1}{a_2} = \frac{b_1}{b_2}$

* III EXACT DIFFERENTIAL EQUATION :-

Let $M(x,y)\,dx + N(x,y)\,dy = 0$ be a first order first degree differential equation, where M and N are real valued functions for some x and y. Then the eq^n $Mdx + Ndy = 0$ is said to be an exact dif. Eq^n if there exists a function f such that $d(f(x,y)) = Mdx + Ndy$

$$\Rightarrow \frac{\partial f}{\partial x}dx + \frac{\partial f}{\partial y}dy = Mdx + Ndy$$

$$\text{(OR)} \quad \frac{\partial f}{\partial x} = M \text{ and } \frac{\partial f}{\partial y} = N.$$

* NOTE: Here $\frac{\partial f}{\partial x} = M$, $\frac{\partial f}{\partial y} = N$.

$$\Rightarrow \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial M}{\partial y}, \quad \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial N}{\partial x}$$

Since $\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$, so that $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

Hence $Mdx + Ndy = 0$ is exact $\Leftrightarrow$ $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

This is condition for exactness.

Ex:- (i) $2xy\,dx + x^2\,dy = 0$ is an exact D.Eq.

Here there exists a $f(x,y) = x^2 y$ such that

$$\frac{\partial f}{\partial x} = 2xy = M \quad \text{and} \quad \frac{\partial f}{\partial y} = x^2 = N.$$

(ii) $x\,dy + y\,dx = 0$ is an exact

Here there exists a $f(x,y) = xy$

such that $\frac{\partial f}{\partial x} = y = M$, $\frac{\partial f}{\partial y} = x = N$.

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Partial Derivatives

let $z = f(x,y)$ be a fn of x & y

If we keep y as constant and vary x alone, the z is a fn of x only the derivative of z w.r.t 'x', treating y as constant, is called the P.D. of z. w.r.t 'x' and is denoted by

$$\frac{\partial z}{\partial x},\ f_x,\ D_x f$$

Thus $$\frac{\partial z}{\partial x} = \lim_{\Delta x \to 0} \frac{f(x+\Delta x, y) - f(x,y)}{\Delta x}.$$

Euler's theorem Problems:-

1) If $u = \sin^{-1}\frac{x}{y} + \tan^{-1}\frac{y}{x}$, P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 0$.

2) If $u = \sin^{-1}\frac{x^2y^2}{x+y}$, P.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 3\tan u$.

3) S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} = 2u\log u$, where $u = e^{x^2+y^2}$.

4) If $u = \sin^{-1}\frac{x+2y+3z}{x^8+y^8+z^8}$, find value of $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z}$?

5) If $u = \frac{x}{y+z} + \frac{y}{z+x} + \frac{z}{x+y}$, S.T $x\frac{\partial u}{\partial x} + y\frac{\partial u}{\partial y} + z\frac{\partial u}{\partial z} = 0$

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* GENERAL SOLUTION OF EXACT D.EQ'S :-

* WORKING RULE:-

(i) Let the D.Eq be of the form $Mdx + Ndy = 0$. check the condition for exactness $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$.

(ii) Find $u(x,y) = \int^{x} M\,dx$, where treating y as constant.

(iii) and find $\phi(y) = \int$ (the terms of N not contains x) dy.

(iv) General solution of exact eq^n is

$$u(x,y) + \phi(y) = c \text{, c is arbitary constant.}$$

i-e, $\int^{x} M\,dx + \int$(terms of N not involving x) dy = c

* PROBLEMS :-

1) Solve $(e^y+1)\cos x\,dx + e^y \sin x\,dy = 0$. ——— (1)

Sol: Eq^n of the form $Mdx + Ndy = 0$

$M = (e^y+1)\cos x$, $\quad N = e^y \sin x$

$\frac{\partial M}{\partial y} = e^y \cos x \qquad \frac{\partial N}{\partial x} = e^y \cos x$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given eq^n is exact

Now $u(x,y) = \int^{x} M\,dx = \int^{x} (e^y+1)\cos x = (e^y+1)\sin x$

$\phi(y) = \int$(terms of N not involving x) dy

$= \int 0\,dy$

$= 0$

Hence G. sol^n is of eq (1) is

$u(x,y) + \phi(y) = c$

$\therefore (e^y+1)\sin x = c$

(2) Solve $(1+e^{x/y})\,dx + e^{x/y}(1-\frac{x}{y})\,dy = 0$. — ①

Sol: $M = 1+e^{x/y}$, $N = e^{x/y}(1-\frac{x}{y})$

$\frac{\partial M}{\partial y} = -\frac{x}{y^2}e^{x/y}$, $\frac{\partial N}{\partial x} = e^{x/y}(-\frac{1}{y}) + \frac{1}{y}e^{x/y}(1-\frac{x}{y}) = -\frac{x}{y^2}e^{x/y}$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$

$\therefore$ Given eq ① is an exact.

Now G. solution is $u(x,y) + \phi(y) = c$

i.e, $\int^x M\,dx + \int (\text{terms of } N \text{ not involving } x)\,dy = c$

$\int^x (1+e^{x/y})\,dx + \int 0.\,dy = c$

$\therefore x + y e^{x/y} = c$

(3) Solve $(xe^{xy} + 2y)\frac{dy}{dx} + ye^{xy} = 0$.

Sol: Given $ye^{xy}\,dx + (xe^{xy} + 2y)\,dy = 0$ — ①

$M = ye^{xy}$, $N = xe^{xy} + 2y$

$\frac{\partial M}{\partial y} = xy\,e^{xy} + e^{xy}$, $\frac{\partial N}{\partial x} = xy\,e^{xy} + e^{xy}$

Thus $\frac{\partial M}{\partial y} = \frac{\partial N}{\partial x}$ $\therefore$ eq ① is exact

G. solⁿ is $\int^x (ye^{xy} \cancel{+2xy})\,dx + \int 2y\,dy = c$

$\cancel{y}\frac{e^{xy}}{\cancel{y}} \cancel{- \frac{e^{xy}}{y^2}} + \cancel{2}y^2 = c$

$\therefore e^{xy} + y^2 = c$ $\quad \cancel{(xy-1)e^{xy} + 2xy^3 = cy^2}$

* H.W.P :-

① Solve $(2x-y+1)\,dx + (2y-x-1)\,dy = 0$ $\quad x^2+y^2-xy+x-y=c$

② Solve $\frac{dy}{dx} + \frac{y\cos x + \sin y + y}{\sin x + x\cos y + x} = 0$ $\quad y\sin x + (\sin y + y)x = c$

③ Solve $2xy\,dy - (x^2-y^2+1)\,dx = 0$. $\quad -x^3 + 3x(1-y^2) = c$

④ Solve $(x^3+3xy^2)\,dx + (y^3+3x^2y)\,dy = 0$.

* ⑤ Solve $(x^2-ay)\,dx = (ax-y^2)\,dy$ $\quad \frac{x^3}{3} - axy + \frac{y^3}{3} = c$ $\quad x^3 - 3axy + y^3 = 3c$

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* **EQUATIONS REDUCIBLE TO EXACT FORM:**

* **INTEGRATING FACTOR :-** Let $M(x,y)dx + N(x,y)dy = 0$ be not an exact difⁿ eqⁿ. If $Mdx + Ndy = 0$ (i) can be made exact by multiplying it with a suitable function $u(x,y) \neq 0$ then $u(x,y)$ is called an integrating factor of $Mdx + Ndy = 0$.

Eg:- let $ydx - xdy = 0$. —— ①

Here $M = y$, $N = -x$

$\frac{\partial M}{\partial y} = 1$, $\frac{\partial N}{\partial x} = -1$

Thus $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

∴ given eqⁿ ① is not exact

Now multiply the given eqⁿ with $\frac{1}{x^2}$, so that

we get $\frac{y}{x^2}dx - \frac{1}{x}dy = 0$ —— ②

$M_1 dx + N_1 dy = 0$

Here $M_1 = \frac{y}{x^2}$, $N_1 = \frac{-1}{x}$

$\frac{\partial M_1}{\partial y} = \frac{1}{x^2}$, $\frac{\partial N_1}{\partial x} = \frac{1}{x^2}$

∴ $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

∴ Eqⁿ ② is exact.

Hence $\frac{1}{x^2}$ is an integrating factor of eq ①.

* **METHODS TO FIND INTEGRATING FACTORS AND SOLUTIONS OF NON-EXACT EQUATIONS :-**

* **METHOD-I (BY INSPECTION):**

With some experience integrating factors can be found by inspection from the following differentials.

(i) $d(xy) = xdy + ydx$

(ii) $d\left(\frac{x}{y}\right) = \frac{ydx - xdy}{y^2}$

(iii) $d\left(\frac{y}{x}\right) = \frac{xdy - ydx}{x^2}$

(iv) $d\left(\frac{e^x}{y}\right) = \frac{ye^x dx - e^x dy}{y^2}$ ....

PROBLEMS:-

(1) Solve $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$

Sol: Given $y\,dx - x\,dy + 3x^2y^2e^{x^3}\,dx = 0$

Dividing by y^2

$$\frac{y\,dx - x\,dy}{y^2} + 3x^2e^{x^3}\,dx = 0$$

$$\int d\left(\frac{x}{y}\right) + \int d\left(e^{x^3}\right) = 0$$

$$\frac{x}{y} + e^{x^3} = c$$

(2) Solve $y(2x^2y + e^x)\,dx = (e^x + y^3)\,dy$

Sol: Given $2x^2y^2\,dx + e^xy\,dx - e^x\,dy - y^3\,dy = 0$

$2x^2y^2\,dx + e^xy\,dx - e^x\,dy - y^3\,dy = 0$

Dividing by y^2

$$2x^2\,dx + \frac{e^xy\,dx - e^x\,dy}{y^2} - y\,dy = 0$$

$$\int 2x^2\,dx + \int d\left(\frac{e^x}{y}\right) - \int y\,dy = 0$$

$$\frac{2x^3}{3} + \frac{e^x}{y} - \frac{y^2}{2} = c$$

* H.W.P :-

(1) Solve $(x^2 - ay)\,dx = (ax - y^2)\,dy$ $x^3 + y^3 - 3axy = c$

(2) Solve $y(2xy + e^x)\,dx - e^x\,dy = 0$. $\frac{x^2}{2} + \frac{e^x}{y} = c$

3) what is the I.F of $\frac{y\,dx - x\,dy}{y^2}$?

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METHOD-2 :-

If $M(x,y)\,dx + N(x,y)\,dy = 0$ is a homogeneous D.Eqn. and not exact. Then $\frac{1}{Mx+Ny}$ $(Mx+Ny \neq 0)$ is an integrating factor of $M dx + N dy = 0$.

Now multiply the given eqn by I.F $\frac{1}{Mx+Ny}$, then that eqn can be transformed to exact form, now solve this eqn. we get a required G.Soln. to the given eqn.

* PROBLEMS :-

(1) Solve $x^2y\,dx - (x^3+y^3)\,dy = 0$.

Sol:- Given $x^2y\,dx - (x^3+y^3)\,dy = 0$ —— ①

$M dx + N dy = 0$

$M = x^2y \qquad N = -(x^3+y^3)$

$\frac{\partial M}{\partial y} = x^2 \qquad \frac{\partial N}{\partial x} = -3x^2$

$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

So that, Given eq ① is not exact.

clearly, it is H.D.Eqn.

Now $Mx+Ny = x^2y.x + (-x^3-y^3)y = -y^4 \neq 0$

$I.F = \frac{1}{Mx+Ny} = \frac{-1}{y^4}$

multiplying eq ① by $-\frac{1}{y^4}$, we get

$-\frac{x^2}{y^3}dx + \left(\frac{x^3}{y^4} + \frac{1}{y}\right)dy = 0$ —— ②

comparing with $M_1dx + N_1dy = 0$

$M_1 = -\frac{x^2}{y^3}, \quad N_1 = \frac{x^3}{y^4} + \frac{1}{y}$

$\frac{\partial M_1}{\partial y} = 3\frac{x^2}{y^4}, \quad \frac{\partial N_1}{\partial x} = \frac{3x^2}{y^3}$

Thus $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$

$\therefore$ Eqn ② is exact.

G. soln of ②, is, $\int^x M_1\,dx + \int(\text{terms of } N_1 \text{ not involve } x)$

$$\int^x -\frac{x^2}{y^3}\,dx + \int \frac{1}{y}\,dy = c$$

$$-\frac{x^3}{3y^3} + \log y = c$$

$\therefore$ which is also G. soln of eq ①.

* H.W.P:-

(1) solve $xy\,dx - (x^2+2y^2)\,dy = 0$ $\quad 4y^2\log y - x^2 = cy^2$

(2) Solve $(x^2y - 2xy^2)\,dx - (x^3 - 3x^2y)\,dy = 0.$ $\quad \frac{x}{y} + \log\frac{y^3}{x^2} = c$

* METHOD - 3:-

If the equation $M\,dx + N\,dy = 0$ is of the form $y f(xy)\,dx + x g(xy)\,dy = 0$ and is not exact. then $\frac{1}{Mx - Ny}$ $(Mx - Ny \neq 0)$ is an integrating factor of $M\,dx + N\,dy = 0$.

Now multiply the given eqn by $\frac{1}{Mx-Ny}$, then that eqn can be transformed to exact form, now solving this eqn we get a required G. soln. to given equation.

PROBLEMS:-

1) Solve $y(x^2y^2 + 2)\,dx + x(2 - 2x^2y^2)\,dy = 0.$ —— ①

Sol: Given $M = x^2y^3 + 2y$, $N = 2x - 2x^3y^2$

$$\frac{\partial M}{\partial y} = 3x^2y^2 + 2, \quad \frac{\partial N}{\partial x} = 2 - 6x^2y^2$$

Thus $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Given eq ① is not exact

and it is in the form $y f(xy)\,dx + x\,g(xy)\,dy = 0$.

Now $Mx - Ny = (x^2y^3 + 2y)x - (2x - 2x^3y^2)y$

$= x^3y^3 + 2xy - 2xy + 2x^3y^3$

$= 3x^3y^3 \neq 0$

$$I.F = \frac{1}{Mx - Ny} = \frac{1}{3x^3y^3}$$

Multiplying eq ① by $I.F = \frac{1}{3x^3y^3}$, we get

$$\frac{x^2y^2+2}{3x^3y^2}\,dx + \frac{2-2x^2y^2}{3x^2y^3}\,dy = 0$$

$$\left[\frac{1}{3x} + \frac{2}{3x^3y^2}\right]dx + \left[\frac{2}{3x^2y^3} - \frac{2}{3y}\right]dy = 0 \quad \text{—②}$$

$M_1\,dx + N_1\,dy = 0$

we have $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$.

$\therefore$ Eq^n ② is exact

G. Sol^n, $\int^x \left(\frac{1}{3x} + \frac{2}{3x^3y^2}\right)dx + \int -\frac{2}{3y}\,dy = 0$

$\therefore \frac{1}{3}\log x + \frac{1}{3x^2y^2} - \frac{2}{3}\log y = c$

* H.W.P :-

① Solve $y(1+xy)\,dx + x(1-xy)\,dy = 0$ $\qquad \log\frac{x}{y} - \frac{1}{xy} = c$

② Solve $(x^2y^2 + xy + 1)y\,dx + (x^2y^2 - xy + 1)x\,dy = 0$ $\qquad xy - \frac{1}{xy} + \log\frac{x}{y} = c$

③ Solve $(xy\sin xy + \cos xy)y\,dx + (xy\sin xy - \cos xy)x\,dy = 0$.

$x\sec xy = cy$

* METHOD- 4 :-

The eqⁿ ~~The~~ $Mdx + Ndy = 0$ is not exact. If there ex* a continuous single variable function $f(x)$ such that $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = f(x)$. then $e^{\int f(x)dx}$ is an I.F of $Mdx + Ndy = 0$.

Now multiplying the given eqⁿ by I.F $e^{\int f(x)dx}$, then that eqⁿ can be transformed to exact form. now solving this equation. we get a required a G. Solⁿ to the given equation.

* PROBLEMS :-

(1) Solve $2xy\,dy - (x^2+y^2+1)\,dx = 0$. —— ①

Sol: Given $N = 2xy$, $M = -x^2 - y^2 - 1$

$$\frac{\partial N}{\partial x} = 2y, \quad \frac{\partial M}{\partial y} = -2y$$

Thus $\frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}$

$\therefore$ Eqⁿ ① is not exact.

We have $\frac{1}{N}\left[\frac{\partial M}{\partial y} - \frac{\partial N}{\partial x}\right] = \frac{1}{2xy}\left[-2y - 2y\right] = -\frac{2}{x} = f(x)$

$$\therefore \text{I.F} = e^{\int f(x)dx} = e^{\int -\frac{2}{x}dx} = e^{-2\log_e x} = x^{-2} = \frac{1}{x^2}$$

Multiplying eq ① with I.F $= \frac{1}{x^2}$, we get

$$-\left(1 + \frac{y^2}{x^2} + \frac{1}{x^2}\right)dx + \frac{2y}{x}dy = 0 \quad \text{—— ②}$$

$$M_1 dx + N_1 dy = 0$$

Verify. $\frac{\partial M_1}{\partial y} = \frac{\partial N_1}{\partial x}$. eq ② is exact.

$\therefore$ G. solⁿ of eq ②, $\int^x M_1 dx + \int(\text{terms of } N_1 \text{ not involving } x)\,dy = c$

$$\int^x \left(-1 - \frac{y^2}{x^2} - \frac{1}{x^2}\right)dx + \int 0\,dy = c$$

$$\therefore -x + \frac{y^2}{x} + \frac{1}{x} = c$$ is G.S of eq ①.

* H.W.P :-

1) Solve $(3xy - 2ay^2)\,dx + (x^2 - 2axy)\,dy = 0$ $\quad x^3y - ax^2y^2 = c$

2) Solve $(x^2+y^2+x)\,dx + xy\,dy = 0$ $\quad 3x^4 + 6x^2y^2 + 4x^3 = c$

3) Solve $(x^2+y^2)\,dx - 2xy\,dy = 0$ $\quad x^2 - y^2 = cx$

*4) Solve $\left(y + \frac{y^3}{3} + \frac{x^2}{2}\right)dx + \frac{1}{4}\left[x + xy^2\right]dy = 0$ $\quad 3x^4y + x^4y^3 + x^6 = c$

(5) Solve $(x^3 - 2y^2)\,dx + 2xy\,dy = 0$ $\quad x^3 + y^2 = cx^2$

* METHOD-5 :-

The equation $Mdx + Ndy = 0$ is not exact. If there exists a continuous single variable function $g(y)$ such that $\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = g(y)$ then $e^{\int g(y)dy}$ is an I.F of $Mdx + Ndy = 0$.

Now multiply the given eqn with I.F $= e^{\int g(y)dy}$, then that eqn can be transformed to exact; now this ~~eqn~~ eqn solving this equation, we get a required G. soln to given eqn.

* PROBLEMS :-

1) Solve $(y^4 + 2y)\,dx + (xy^3 + 2y^4 - 4x)\,dy = 0$. ——(1)

Sol: Given $M = y^4 + 2y$, $\quad N = xy^3 + 2y^3 - 4x$

$$\frac{\partial M}{\partial y} = 4y^3 + 2, \quad \frac{\partial N}{\partial x} = y^3 - 4$$

$$\therefore \frac{\partial M}{\partial y} \neq \frac{\partial N}{\partial x}.$$

$\therefore$ Eqn is not exact.

Now $$\frac{1}{M}\left[\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right] = \frac{1}{y^4+2y}\left[y^3 - 4 - 4y^3 - 2\right] = \frac{-3[y^3+2]}{y[y^3+2]}$$

$$= \frac{-3}{y} = g(y)$$

$$\therefore \text{I.F} = e^{\int g(y)\,dy} = e^{\int -\frac{3}{y}dy} = e^{-3\log_e y} = \frac{1}{y^3}$$

Now multiplying eq (1) with I.F $= \frac{1}{y^3}$, we get

$$\left(y+\frac{2}{y^2}\right)dx+\left(x+2y-\frac{4x}{y^3}\right)dy=0 \quad \text{—} ②$$

$$M_1\,dx+N_1\,dy=0$$

$$M_1=y+\frac{2}{y^2}, \quad N_1=x+2y-\frac{4x}{y^3}$$

verify $\frac{\partial M_1}{\partial y}=\frac{\partial N_1}{\partial x}$, eq ② is exact.

G. solⁿ of eq ②, $\int^{x}\left(y+\frac{2}{y^2}\right)dx+\int 2y\,dy=c$

$$\therefore xy+2\frac{x}{y^2}+y^2=c$$

which is also G.S. of eq ①.

H.W.P:-

*① Solve $y(xy+e^x)\,dx-e^x\,dy=0$ $\quad \frac{x^2}{2}+\frac{e^x}{y}=c$

② Solve $(xy^2-x^2)\,dx+(3x^2y^2+x^2y-2x^3)\,dy=0$ $\quad e^{6y}\left(\frac{x^2y^2}{2}-\frac{x^3}{3}\right)=c$

③ Solve $(3x^2y^4+2xy)\,dx+(2x^3y^3-x^2)\,dy=0$ $\quad x^3y^3+x^2=cy$

*④ Solve $(xy^3+y)\,dx+2(x^2y^2+x+y^4)\,dy=0$

$$\frac{x^2y^4}{2}+y^2x+\frac{2y^6}{6}=c.$$

* IV LINEAR DIFFERENTIAL EQUATIONS OF FIRST ORDER :-

An equation of the form $\frac{dy}{dx} + P(x)y = Q(x)$, where P and Q are either constants (or) functions of x only is, called a L.D.Eq of first order in y.

* SOLUTION OF L.D.Eq :-

* Working rule :-

(i) Rewrite the given eqn in the form $\frac{dy}{dx} + P(x)y = Q(x)$.

(ii) Finding the I.F $= e^{\int P(x)dx}$ of given eqn.

(iii) G. Soln of L.D.Eq is

$$y\, e^{\int Pdx} = \int Q(x) \cdot e^{\int Pdx}\, dx + c$$

(OR)

$$y(IF) = \int Q(I.F)\, dx + c.$$

* ANOTHER FORM OF L.D.Eq :-

An equation of the form $\frac{dx}{dy} + P_1(y)x = Q_1(y)$, where P_1 and Q_1 are either constants (or) functions of y only is called L.D.Eq of first order in x.

SOLUTION

write the I.f $= e^{\int P_1(y)\,dy}$

G. Soln is $x(IF) = \int Q_1(IF)\,dy + c$

* PROBLEMS :-

1) Solve the differential equation $x\frac{dy}{dx}+y=\log x$

Sol: Given $x\frac{dy}{dx}+y=\log x$

Dividing by 'x', $\frac{dy}{dx}+\frac{1}{x}y=\frac{1}{x}\log x$ — ①

It is in the form $\frac{dy}{dx}+P(x)y=Q(x)$

where $P(x)=\frac{1}{x}$, $Q(x)=\log x\cdot\frac{1}{x}$

$\therefore$ $I.F = e^{\int P(x)dx} = e^{\int\frac{1}{x}dx} = e^{\log_e x} = x$

$\therefore$ G.Solⁿ of ① is $y(IF)=\int Q(x).(IF)dx+c$

$yx=\int\frac{1}{\cancel{x}}\log x.\cancel{x}\,dx+c$

$\therefore xy = x(\log x-1)+c$

2) Solve $\frac{dy}{dx}+y=x^2$

Sol:- $P=1$, $Q=x^2$

$I.F = e^{\int 1dx} = e^x$

G. Solⁿ is $y.(IF)=\int Q(IF)dx+c$

$ye^x=\int x^2e^x dx+c$

$= x^2e^x-\int 2xe^x dx+c$

$= x^2e^x-2xe^x+2e^x+c$

$\therefore ye^x = e^x(x^2-2x+2)+c$

3) Solve $(1-x^2)\frac{dy}{dx}+2xy=x\sqrt{1-x^2}$

Sol: Given L.D.Eq $\frac{dy}{dx}+\frac{2x}{1-x^2}y=\frac{x}{\sqrt{1-x^2}}$

where $P=\frac{2x}{1-x^2}$, $Q=\frac{x}{\sqrt{1-x^2}}$

$I.F = e^{\int\frac{2x}{1-x^2}dx} = e^{-\log_e(1-x^2)} = \frac{1}{1-x^2}$

G.Sol^n is $y \cdot \frac{1}{1-x^2} = \int \frac{x}{\sqrt{1-x^2}} \cdot \left(\frac{1}{1-x^2}\right) dx + c$

$= \int \frac{x}{(1-x^2)^{3/2}} dx + c$

put $1-x^2 = t$

$x\,dx = \frac{-1}{2} dt$

$= \int \frac{1}{t^{3/2}} \frac{-1}{2} dt + c$

$= \frac{-1}{2} \cdot \frac{t^{-1/2}}{-1/2} + c$

$\therefore y \frac{1}{1-x^2} = \frac{1}{\sqrt{1-x^2}} dx + c$

* ~~H.W.P~~ :-

4) Solve $(1+y^2)\,dx = (\tan^{-1}y - x)\,dy$

Sol: write in the form $\frac{dx}{dy} = \frac{\tan^{-1}y}{1+y^2} - \frac{x}{1+y^2}$

$\frac{dx}{dy} + \frac{1}{1+y^2} x = \frac{\tan^{-1}y}{1+y^2}$ —①

where $P_1(y) = \frac{1}{1+y^2}$, $Q_1 = \frac{\tan^{-1}y}{1+y^2}$

$I.F = e^{\int P_1(y)dy} = e^{\int \frac{1}{1+y^2} dy} = e^{\tan^{-1}y}$

G.Sol^n of ① is $x(IF) = \int Q_1(y).(IF)\,dy + c$

$x.\, e^{\tan^{-1}y} = \int \frac{\tan^{-1}y}{1+y^2} e^{\tan^{-1}y} dy + c$

put $\tan^{-1}y = t$

$\frac{1}{1+y^2} dy = dt$

$= \int t e^t\, dt + c$

$= te^t - 1e^t + c$

$= (t-1)e^t + c$

$\therefore x \tan^{-1}y = (\tan^{-1}y - 1) e^{\tan^{-1}y} + c$

(5) H.W.P:-

$ye^{kx} = x + c$

1) Solve $y' + ky = e^{-kx}$

2) Solve $y' + y = e^{e^x}$ $\qquad ye^x = e^{e^x} + c$

3) Solve $(1-x^2)\frac{dy}{dx} + xy = ax$ $\qquad y = ax + c\sqrt{1-x^2}$

4) Solve $(x+2y^3)\frac{dy}{dx} = y$ $\qquad x = y^3 + cy$

5) Solve $(x+y+1)\frac{dy}{dx} = 1.$ $\qquad xe^{-y} = -e^{-y}(y+2) + c$

6) Solve $x\log x \frac{dy}{dx} + y = 2\log x$ $\qquad y\log x = (\log x)^2 + c$

*7) Find the I.F of $x\frac{dy}{dx} + y = \log x.$

*8) Solve $\frac{dx}{dy} + x = y^2.$ $\qquad x - y^2 + 2y - 2 = ce^{-y}$

* BERNOULLI'S EQUATION:-

An equation of the form $\frac{dy}{dx} + Py = Qy^n$ is called Bernoulli's equation, where P and Q are either constants (or) functions of x only and n is real no' such that $n \neq 0, n \neq 1$.

If $n = 0$, (or) $n = 1$, the Bernoulli's eqn becomes L.D.Eqn.

* SOLUTION OF BERNOULLI'S EQn:-

Given B.Eqn $\frac{dy}{dx} + Py = Qy^n$, $n \neq 0, n \neq 1$. — (1)

multiplying y^{-n} on B.S

$$y^{-n}\frac{dy}{dx} + Py^{1-n} = Q. \quad — (2)$$

put $y^{1-n} = t$

$$\Rightarrow (1-n)y^{-n}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-n}\frac{dy}{dx} = \frac{1}{1-n}\frac{dt}{dx}$$

From (2), $\frac{dt}{dx} + (1-n)Pt = (1-n)Q$ — (3)

which is a L.D.E in 't'

find soln of eq (3), again put $t = y^{1-n}$.

* ANOTHER FORM OF BERNOULLI's Eqn :-

An equation of the form $\frac{dx}{dy} + P_1(y)x = Q(y)x^n$, where P_1 and Q_1 are either constants (or) functions of y only and n [$n \neq 0, n \neq 1$] is a real number.

SOLUTION :-

$$\frac{dx}{dy} + P_1 x = Q_1 x^n , \quad n \neq 0, n \neq 1$$

$$x^{-n}\frac{dx}{dy} + P_1 x^{1-n} = Q_1$$

put $x^{1-n} = t$.

⋮

* PROBLEMS :-

(1) Solve $x\frac{dy}{dx} + y = x^3y^6$

Sol: Given $\frac{dy}{dx} + \frac{1}{x}y = x^2y^6$ —— ①

which is B. Eqn.

multiplying by y^{-6}

$$y^{-6}\frac{dy}{dx} + \frac{1}{x}y^{-5} = x^2 \quad —— ②$$

put $y^{-5} = t$

$$-5y^{-6}\frac{dy}{dx} = \frac{dt}{dx}$$

$$y^{-6}\frac{dy}{dx} = -\frac{1}{5}\frac{dt}{dx}$$

From eq ②, $\frac{dt}{dx} - \frac{5}{x}t = -5x^2$ —— ③

which is L.D.E

$$P = -\frac{5}{x}, \quad Q = -5x^2$$

$$IF = e^{\int -\frac{5}{x}dx} = e^{-5\log_e x} = \frac{1}{x^5}$$

G.S of ③, $t(IF) = \int Q(IF)\,dx + c$

$$t \cdot \frac{1}{x^5} = \int -5x^2\frac{1}{x^5}dx + c$$

$$t\frac{1}{x^5} = +\frac{5}{2x^2} + c$$

put $t = y^{-5}$ ∴ $y^{-5}\frac{1}{x^5} = \frac{5}{2x^2} + c \Leftrightarrow ∴ \frac{1}{y^5} = \frac{5}{2}x^3 + cx^5$

2) Solve the following diff^n. eq^n by reducing to a L.D.Eq

$$\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$$

Sol: Given $\frac{dy}{dx} = \frac{y}{x+\sqrt{xy}}$

$$y\frac{dx}{dy} = x + \sqrt{x}\sqrt{y}$$

$$\frac{dx}{dy} = \frac{1}{y}x + \frac{\sqrt{x}}{\sqrt{y}}$$

$$\frac{dx}{dy} - \frac{1}{y}x = \sqrt{x}\cdot\frac{1}{\sqrt{y}} \Leftrightarrow \frac{dx}{dy} - \frac{1}{y}x = \frac{1}{\sqrt{y}}\cdot x^{+1/2} \quad \text{—①}$$

which is B-Eq^n

multi by $x^{-1/2}$

$$x^{-1/2}\frac{dx}{dy} - \frac{1}{y}x^{1/2} = \frac{1}{\sqrt{y}} \quad \text{—②}$$

put $x^{1/2} = t$

$$\frac{1}{2\sqrt{x}}\frac{dx}{dy} = \frac{dt}{dy}$$

$$x^{-1/2}\frac{dx}{dy} = 2\frac{dt}{dy}$$

$\therefore$ From ②, $\cancel{2}\frac{dt}{dy} - \frac{1}{2y}t = \frac{1\cancel{2}}{2\sqrt{y}}$ —③

L.D.Eq in 't'

$$P_1 = \frac{-1}{2y},\ Q_1 = \frac{2}{\sqrt{y}}$$

$$\text{I.F} = e^{\int -\frac{1}{2y}dy} = e^{-\frac{1}{2}\log_e y} = \frac{1}{\sqrt{y}}$$

G. soln of eq ③, $t\cdot\frac{1}{\sqrt{y}} = \int \frac{\cancel{2}}{2\sqrt{y}}\cdot\frac{1}{\sqrt{y}}dy + c$

$$t\cdot\frac{1}{\sqrt{y}} = \frac{1}{2}\int\frac{1}{y}dy + c$$

put $t = x^{1/2}$; $\quad x^{1/2}\frac{1}{\sqrt{y}} = \frac{1}{2}\log y + c$

$$\therefore \sqrt{\frac{x}{y}} = \frac{1}{2}\log y + c$$

which G.S of eq ①.

3 Solve $\dfrac{dy}{dx} - y\tan x = -y^2 \sec x$. — (1)

B. Eqn

Sol:

~~Divide~~ multiplying by y^{-2}

$$y^{-2}\frac{dy}{dx} - y^{-1}\tan x = -\sec x. \quad \text{— (2)}$$

put $y^{-1} = t \Rightarrow -\dfrac{1}{y^2}\dfrac{dy}{dx} = \dfrac{dt}{dx}$

$$y^{-2}\frac{dy}{dx} = -\frac{dt}{dx}$$

From (2),

$$\frac{dt}{dx} + \tan x \cdot t = \sec x. \quad \text{— (3)}$$

L.D. Eqn

$$P = \tan x, \quad Q = \sec x.$$

$$I.F = e^{\int \tan x\, dx} = e^{\log_e \sec x} = \sec x.$$

G.S of eq (3), $\quad t \sec x = \int \sec x \cdot \sec x \, dx + c$

$$t \cdot \sec x = \int \sec^2 x \, dx + c$$

$$t \cdot \sec x = \tan x + c$$

G.S of eq (1), put $t = y^{-1}$

$$\therefore \frac{1}{y}\sec x = \tan x + c.$$

* H.W.P :-

(1) Solve $x\dfrac{dy}{dx} + y = x^2 y^6$ $\qquad 3 = 5x^2y^5 + 3cx^5y^5$

2) Solve $\dfrac{dy}{dx} + \dfrac{y}{x} = y^2 x \sin x$ $\qquad \dfrac{1}{y} = x\cos x + cx$

3) Solve $\dfrac{dy}{dx}(x^2y^3 + xy) = 1.$ $\qquad \dfrac{1}{x}e^{y^2/2} = (2-y^2)e^{y^2/2} + c$

* 4) Solve $y' + 2y = y^2$ $\qquad 2 = y[1 + 2ce^{2x}]$

* D.EQ'S REDUCIBLE TO L.D.Eq^n BY SUBSTITUTION :-

(1) Solve $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$

So

Sol:- Given $\frac{dy}{dx} + x \sin 2y = x^3 \cos^2 y$ —— ①

Dividing by $\cos^2 y$.

$$\sec^2 y \frac{dy}{dx} + x \cdot \frac{2 \sin y \cos y}{\cos^2 y} = x^3$$

$$\sec^2 y \frac{dy}{dx} + 2x \cdot \tan y = x^3 \quad —— ②$$

put $\tan y = t$

$$\sec^2 y \frac{dy}{dx} = \frac{dt}{dx}$$

From ②, $\frac{dt}{dx} + 2x \cdot t = x^3$ —— ③

L.D.Eq in t.

$P = 2x, \quad Q = x^3$

$$I.F = e^{\int 2x dx} = e^{x^2}$$

G.S of ③, $t e^{x^2} = \int x^3 e^{x^2} dx + c$

put $x^2 = u$

$x dx = \frac{1}{2} du$

$$= \int u e^u \cdot \frac{1}{2} du + c$$

$$= \frac{1}{2}(u-1)e^u + c$$

$$t \cdot e^{x^2} = \frac{1}{2}(x^2-1)e^{x^2} + c$$

$\therefore$ G.S of eq ①, put $t = \tan y$

$$\therefore \tan y \cdot e^{x^2} = \frac{1}{2}(x^2-1)e^{x^2} + c$$

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(2) Solve $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$

Sol: Given $\frac{dy}{dx} - \frac{\tan y}{1+x} = (1+x)e^x \sec y$ —— ①

Dividing by $\sec y$

$$\cos y \frac{dy}{dx} - \frac{1}{1+x} \cdot \sin y = (1+x)e^x \quad —— ②$$

put $\sin y = t \Rightarrow \cos y \frac{dy}{dx} = \frac{dt}{dx}$

From eq ②, $\frac{dt}{dx} - \frac{1}{1+x} t = (1+x)e^x$ —— ③

L.D.Eq in t

$$P = \frac{-1}{1+x}, \quad Q = (1+x)e^x$$

$$I.F = e^{\int -\frac{1}{1+x} dx} = e^{-\log_e(1+x)} = \frac{1}{1+x}$$

G.S of eq ③, $t \cdot \frac{1}{1+x} = \int (1+x)e^x \cdot \frac{1}{1+x} dx + c$

$$t \frac{1}{1+x} = e^x + c$$

G.S of ①, put $t = \sin y$, $\frac{\sin y}{1+x} = e^x + c$ //

* H.W.P:-

1) Solve $x^2 \frac{dy}{dx} = e^y - x$ $\quad e^{-y} \frac{1}{x^2} = \frac{1}{2x^2} + c$

2) Solve $2y \cos y^2 \frac{dy}{dx} - \frac{2}{x+1} \sin y^2 = (x+1)^3$ $\quad \sin y^2 = \frac{(x+1)^4}{2} + c(x+1)^2$

* 3) solve $\tan y \frac{dy}{dx} + \tan x = \cos y \cos^2 x$ $\quad \sec x \sec y = \sin x + c$

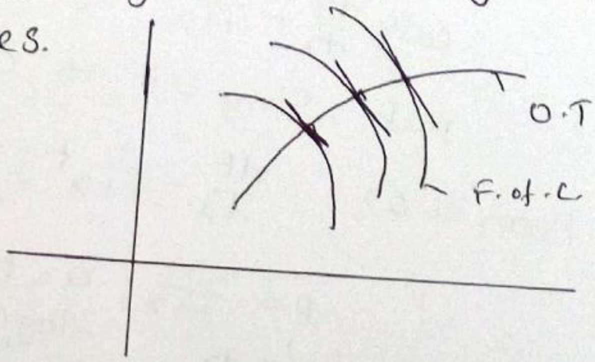
* 4) Solve $x^3 \sec^2 y \frac{dy}{dx} + 3x^2 \tan y = \cos x$

$(x^3 \tan y = \sin x + c)$

* APPLICATIONS OF D. EQ'S OF FIRST ORDER :-

(1) ORTHOGONAL TRAJECTORIES :

* A curve which cuts every member of a given family of curves at a right angle is an 'orthogal trajectory' of the given family curves.



If every curve of a family (I) of curves cuts every member of another family (II) of curves at a right angles. then (I) is said to be a family of O.T of (II).

If (I) is a family of orthogonal trajectories of (II), then (II) is also a family of O.T of (I).

* O.T IN CARTESIAN CO-ORDINATES :-

* WORKING RULE :-

(I) let $f(x,y,c)=0$ — ① be a family of curve, c is parameter.

(ii) Diffⁿ eq ①, w.r.t 'x' and eliminate c, we get the D Eqⁿ $F(x,y,y')=0$ — ② of given family of curves.

(iii) Replace $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in eq ②, then the D. Eqⁿ of the family of orthogonal trajectories is $F(x,y,-\frac{dx}{dy})=0$. — ③

(iv) Solve the eqⁿ ③ to get the eqn of the family of O.T of eq ①.

* PROBLEMS :-

1) Find the O.T. of the family of curve $ay^2 = x^3$, where 'a' is a parameter.

Sol: Given family of curve $ay^2 = x^3$ — ①

Dif^n eq ② w.r.t x

$$a\,2y\frac{dy}{dx} = 3x^2 \text{ — ②}$$

eliminate 'a', put $a = \frac{x^3}{y^2}$ in eq ②.

$$\frac{x^3}{y^2}\cdot 2y\frac{dy}{dx} = 3x^2$$

$$\frac{dy}{dx} = \frac{3}{2}\frac{y}{x} \text{ — ③}$$

which is D.Eq^n of eq ①.

Now replacing $\frac{dy}{dx}$ by $-\frac{dx}{dy}$ in ③

$$-\frac{dx}{dy} = \frac{3}{2}\frac{y}{x}.$$

$$\int 2x\,dx = \int -3y\,dy$$

$$\therefore x^2 + \frac{3y^2}{2} = c.$$

which is required O.T of eq ①.

* H.W.P :-

1) Find the O.T of family of curves $y = ax$. $x^2+y^2 = c^2$

2) " " " " $x^2+y^2 = a^2$ $y = cx$

3) " " " " $x^{2/3}+y^{2/3} = a^{2/3}$ $x^{4/3}-y^{4/3} = c$

4) " " " " $x^2+cy^2 = 1$. $\frac{y^2}{2}+\frac{x^2}{2} = \log x + c$

5) " " " " $y^2 = 4ax$.

6) " " " " $x^2 = 4ay$.

* SELF ORTHOGONAL :-

If the diff^n eq^n of a given family of curve is same as the diff^n. eq^n of the family of O.T. Then the given family of curve is Self orthogonal.

* Eg:

1) Show that the system of Confocal conics $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$ where λ is parameter is Self orthogonal.

Sol: Given $\frac{x^2}{a^2+\lambda} + \frac{y^2}{b^2+\lambda} = 1$, λ is parameter —— (1)

Dif^n. (1) w.r.t x

$$\frac{2x}{a^2+\lambda} + \frac{2y}{b^2+\lambda}\frac{dy}{dx} = 0 \Rightarrow x(b^2+\lambda) + yp(a^2+\lambda) = 0$$

$$\lambda(x+yp) = -(xb^2+a^2yp)$$

$$\lambda = -\frac{(b^2x+a^2yp)}{x+yp}$$

$$\therefore a^2+\lambda = \frac{(a^2-b^2)x}{x+yp}, \quad b^2+\lambda = \frac{-(a^2-b^2)yp}{x+yp}$$

Now eliminate λ, from eq (1)

$$\frac{(x+yp)(x)}{(a^2-b^2)} - \frac{(x+yp)y}{(a^2-b^2)p} = 1$$

$$(x+yp)\left(x-\frac{y}{p}\right) = a^2-b^2$$

i.e, $\left(x+y\frac{dy}{dx}\right)\left(x-\frac{y}{(dy/dx)}\right) = a^2-b^2$ —— (2)

which D. Eq of (1).

Replacing dy/dx by $-dx/dy$ in (2), we get

$$\left(x-\frac{y}{dy/dx}\right)\left(x+y\frac{dy}{dx}\right) = a^2-b^2 \quad \text{—— (3)}$$

which is D.Eq^n of O.T

observe that eq (2), eq (3) are same.

Hence Given family of curve (1) is S.O.

H.W.P.

1) Show that system of Parabolas $y^2 = 4a(x+a)$ is S.O.

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* O.T. IN POLAR CO-ORDINATES :-

(i) let $f(r,\theta,c)=0$ — ① be a given family of curve in polar co-ordinats, c is parameter.

(ii) Diffⁿ eq ①, w.r.t 'θ' and eliminate c, we get the diffⁿ. eqⁿ $F(r,\theta,\frac{dr}{d\theta})=0$ — ② of eq ①.

(iii) Replace $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in eq ②, we get the diffⁿ. eqⁿ $F(r,\theta,-r^2\frac{d\theta}{dr})=0$ — ③ of O.T.

[∵ In polar co-ordinates, $\tan\phi = r\frac{d\theta}{dr}$ and if two curves are orthogonal $r\frac{d\theta}{dr}\times r\frac{d\theta}{dr}=-1$

$$\Leftrightarrow r\frac{d\theta}{dr} = -\frac{1}{r}\frac{dr}{d\theta}$$

$$\Rightarrow \frac{dr}{d\theta} = -r^2\frac{d\theta}{dr}\ ]$$

(iv) Solve the eq ③, to get the equation of O.T of eq ①.

* PROBLEMS :-

1) Find the O.T. of the family of cardioids $r=a(1-\cos\theta)$ where a is parameter.

Sol: Given family of curve $r=a(1-\cos\theta)$ — ①

Dⁿ w.r.t. 'θ'

$$\frac{dr}{d\theta} = a\sin\theta \Rightarrow a = \frac{1}{\sin\theta}\frac{dr}{d\theta} \quad —②$$

eliminate 'a', from ①, ②, we have

$$r = \frac{1}{\sin\theta}(1-\cos\theta)\frac{dr}{d\theta}$$

$$r = \frac{2\sin^2\theta/2}{2\sin\theta/2\cos\theta/2}\frac{dr}{d\theta}$$

$$\frac{dr}{d\theta} = r\cot\theta/2 \quad —③$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$ in eq ③

$$-r^2\frac{d\theta}{dr} = r\cot\frac{\theta}{2}$$

$$\frac{1}{r}dr = -\tan\frac{\theta}{2}\,d\theta$$

$$\int\frac{1}{r}dr = \int -\tan\frac{\theta}{2}\,d\theta$$

$$\log r = +\log\cos\frac{\theta}{2} + \log C$$

$$\log r = \log C.\cos\frac{\theta}{2}$$

$$r = C.\frac{1+\cos\theta}{2}$$

$$\therefore\ r = K(1+\cos\theta) \quad (\because K = \tfrac{C}{2})$$

which is O.T of eq ①

2) Find the O.T of the family of curves $r^n = a^n\cos n\theta$.

Sol: Given $r^n = a^n\cos n\theta$. —— ①

Taking logarithms on B.S

$$n\log r = n\log a + \log\cos n\theta$$

Dn w.r.t 'θ'

$$n\frac{1}{r}\cdot\frac{dr}{d\theta} = \frac{1}{\cos n\theta}\cdot(-\sin n\theta).n \Rightarrow \frac{dr}{d\theta} = -r\tan n\theta \quad —②$$

Replacing $\frac{dr}{d\theta}$ by $-r^2\frac{d\theta}{dr}$

$$-r^2\frac{d\theta}{dr} = -r\tan n\theta \Rightarrow \cot n\theta\, d\theta = \frac{1}{r}dr$$

$$\Rightarrow \int\frac{1}{r}dr = \int\cot n\theta\, d\theta$$

$$\log r = \frac{\log\sin n\theta}{n} + \log c \Rightarrow \log r^n = \log\sin n\theta.c^n$$

$$\therefore r^n = c^n\sin n\theta$$

* H.W.P:-

1) Find the O.T of family of curves $r^2 = a^2\cos\theta$. $r^2 = c^2\sin\theta$

2) " " " $r^n\sin n\theta = a^n$ $r^n\cos n\theta = c^n$

3) " " " $r = c\sin\theta$ $r = a\cos\theta$

(2) NEWTON'S LAW OF COOLING:-

* Statement:- The rate of change of the temperature of a body is proportional to the difference of the temperature of the body and that of the surrounding medium.

i.e, let θ be the temperature of the body at time "t" and θ_0 be the temperature of its surrounding medium.

By Newton's law of cooling

$$\frac{d\theta}{dt} \propto (\theta - \theta_0)$$

i.e, $$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$, k is +ve constant

Solve this which gives the temperature of the body at any time "t".

* PROBLEMS:-

1) If the air is maintained at 20°c and the temperature of the body cools from 100°c to 80°c in 10 min. find the temperature of the body after 20 mints.

Sol: let the temperature of body is "θ"

and temperature of the air $\theta_0 = 20°c$

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0)$$. k is +ve constant

$$\frac{d\theta}{dt} = -k(\theta - 20)$$

$$\int \frac{1}{\theta - 20} d\theta = \int -k\, dt$$

$$\log(\theta - 20) = -kt + \log c$$, c is constant.

$$\log \frac{\theta - 20}{c} = -kt$$

$\therefore \quad \theta - 20^\circ = e^{-kt} \cdot e^{c_1}$, ($\because$ Take $e^{c_1} = c$)

$\theta - 20^\circ = c e^{-kt}$ ——— ①

Given Originally temperature of body at $t = 0$ min is $\theta = 100^\circ c$

$\therefore$ From eq ①, $100^\circ - 20^\circ = c$

$c = 80^\circ$

$\therefore \quad \theta - 20^\circ = 80 e^{-kt}$ ——— ②

when $t = 10$ mts, $\theta = 80^\circ c$

From eq ①, $80^\circ - 20^\circ = 80 e^{-k(10)}$

$60 = 80^\circ e^{-10k}$

$e^{-10k} = \frac{3}{4}$.

When $t = 20$ mts, the body's temperature

From eq ①, $\theta - 20^\circ = 80 \cdot e^{-k(20)}$

$= 80^\circ (e^{-10k})^2$

$= 80^\circ \cdot (\frac{3}{4})^2$

$\theta - 20^\circ = 45^\circ$

$\boxed{\therefore \quad \theta = 65^\circ c}$

2) If the temperature of the air is $30^\circ c$ and the ~~substa~~ substance cools from $100^\circ c$ to $70^\circ c$ in 15 mints. Find when the temperature will be $40^\circ c$.

Sol: let θ be the temperature of substance and temperature of the air $\theta_0 = 30^\circ c$

By Newton's law of cooling,

$$\frac{d\theta}{dt} = -k(\theta - \theta_0), \quad k > 0$$

$$\frac{d\theta}{dt} = -k(\theta - 30^\circ)$$

$$\int \frac{1}{\theta - 30^\circ} d\theta = \int -k\,dt \Rightarrow \log(\theta - 30^\circ) = -kt + \log c$$

$$(\text{or})\ \theta - 30^\circ = c e^{-kt}, \quad c \text{ is constant} \quad \text{—— (1)}$$

Given at $t = 0\,\text{min}$, $\theta = 100^\circ c$

From (1), $100^\circ - 30^\circ = c e^{0}$

$c = 70^\circ c$

and at $t = 15\,\text{mts}$, $\theta = 70^\circ c$

from (1), $\cancel{70^\circ - 30^\circ = 70^\circ e^{-15k}}$ ~~(or)~~

$$\log(70^\circ - 30^\circ) = -k(15) + \log 70^\circ$$

$$\log 40^\circ = -15k + \log 70^\circ$$

$$15k = \log 70^\circ - \log 40^\circ$$

$$k = \frac{1}{15} \log \frac{7}{4}$$

at $\theta = 40^\circ c$, from (1), $\log(40^\circ - 30^\circ) = -\frac{t}{15} \log \frac{7}{4} + \log 70^\circ$

$$-\log 10^\circ + \log 70^\circ = \frac{1}{15} \cdot \log \frac{7}{4} \cdot t$$

$$\log 7 = \frac{1}{15} \log \frac{7}{4} \cdot t$$

$$t = 15 \cdot \frac{\log 7}{\log 7/4}$$

$$= 15 \times 3.48$$

$$t = 52.16 \text{ mts}$$

$\therefore$ The temperature will be $40^\circ c$ after 52.16 mts.

* H.W.P :-

1) If the Surroundings are maintained at $30^\circ c$ and the temperature of body cools from $80^\circ c$ to $60^\circ c$ in 12 mts, find the temperature of body after 24 mts. $\theta = 30.36^\circ c$

2) A body kept in air with temperature $25^\circ c$ cools from $140^\circ c$ to $80^\circ c$ in 20 mints. Find when the body cools down to $35^\circ c$.

$t = 66.2$ mts

(3) LAW OF NATURAL GROWTH (OR) DECAY :

*

Statement:- Let $x(t)$ be the amount of substance at time 't' and let the substance be getting converted chemically. A law of chemical conversion states that the rate of change of amount $x(t)$ of a chemically changing substance is proportional to the amount of the substance available at the time.

i.e, $\frac{dx}{dt} \propto x$ (or) $\frac{dx}{dt} = kx$, $k \in \mathbb{R}$.

* This D.Eqⁿ can also describe in a simple way the population growth, radioactive decay etc.

* If ~~as~~ x increases as t increases we can take

$$\frac{dx}{dt} = kx \quad (k>0)$$

If x decreases as t increases we can take

$$\frac{dx}{dt} = -kx \quad (k>0).$$

* PROBLEM:-

1) In certain culture of bacteria the rate of increase is proportional to the number present. If it is found that the number doubles in 4 hours, how many may be expected at the end of 12 hours?

Sol:- let N be the number of bacteria present at $t=0$.

and N_1 be the number of bacteria at any time $t>0$.

By law of natural growth

$$\frac{dN_1}{dt} = kN_1, \quad k>0$$

$$\int \frac{1}{N_1} dN_1 = \int k\,dt$$

$$\log N_1 = Kt + \log c$$

$$N_1 = ce^{kt} \quad \text{—} ①$$

when $t = 0$ mit, $N_1 = N$.

$\therefore N = C \Rightarrow c = N$.

from given In 4 hrs, bacteria doubles.

i.e, at $t = 4$ hrs; $N_1 = 2N$.

From ①, $2N = N e^{K4}$

$2 = e^{4k}$ (or) $\log 2 = 4k$.

$\therefore K = \frac{1}{4}\log 2$

Now After 12 hrs,

i.e at $t = 12$ hrs,

From ①, $N_1 = N.e^{12k}$

$= N.(e^{4k})^3$

$= N(2)^3$

$\therefore N_1 = 8N$.

$\therefore$ After, end of 12 hours, total bacteria ~~are~~ will be ~~Increases~~ 8 times of Present number.

H.W. P.:-

① Bacteria in a culture grows exponentially so that the initial number has doubled in 3 hrs. How many times the initial number will be present after 9 hrs?

N = 8A