# UNIT- V

## Vector Differential Calculus

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Tangent vector of a curve in space:-

let $\bar{r} = x\hat{i} + y\bar{j} + z\bar{k}$ be the position vector of any point on the curve. Let $x = x(t), y = y(t)$, $z = z(t)$, are continuous & derivable. Then

$\frac{d\bar{r}}{dt}$ is said to a tangent vector to the curve at any point P.

unit vector:- let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be any vector function then unit vector of $\bar{f}$ is define as

$$\bar{e} = \hat{f} = \frac{f_1\bar{i} + f_2\bar{j} + f_3\bar{k}}{\sqrt{f_1^2 + f_2^2 + f_3^2}}$$

Directional Derivative:-

The D.D of a scalar point function $\phi(x,y,z)$ at a point $P(x,y,z)$ in the direction of a unit vector $\bar{e}$ is define as $\nabla\phi.\bar{e}$

i.e $D_{\bar{e}}\phi(x,y,z) = \nabla\phi.\bar{e}$

$\bar{b}$

$\therefore$ D.D $= \nabla\phi.\bar{e}$, where $\bar{e} = \frac{\bar{b}}{|\bar{b}|}$

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# UNIT-V

## VECTOR DIFFERENTIAL CALCULUS

* **Scalar function (field)** :- A scalar function $f(x,y,z)$ is a function defined at each point in certain domain D in a space. It's value is real and depends on the point $P(x,y,z)$ in space.

For every point $(x,y,z) \in D$, $f$ has a real value. We say that a scalar field $f$ is defined in D.

Ex:- The distance fn in 3-D space which defines the distance b/w the points $P(x,y,z)$ & $P_0(x_0,y_0,z_0)$

$$f(P) = f(x,y,z) = \sqrt{(x-x_0)^2 + (y-y_0)^2 + (z-z_0)^2}$$

defines a scalar field.

let $\bar{i}, \bar{j}, \bar{k}$ be three mutually $\perp^r$ unit vectors in 3-D space, then the vector function $f(t)$ of a scalar variable $t$ may be expressed in the form $\bar{f}(t) = f_1(t)\bar{i} + f_2(t)\bar{j} + f_3(t)\bar{k}$, where $f_1(t), f_2(t), f_3(t)$ are the real valued functions and are called the components of $\bar{f}(t)$.

* **Vector function (field)** :- A function $\bar{V}(P) = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ defined at each point $P \in D$ is called a vector fn. We say that a vector field is defined in D. In carteasian coordinates, we can write

$$\bar{V}(x,y,z) = f_1(x,y,z)\bar{i} + f_2(x,y,z)\bar{j} + f_3(x,y,z)\bar{k}.$$

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Ex:- the velocity field $v(p)$ defined at any point $p$ on a rotating body defines a vector field.

If the scalar and vector fields depend on time also, then we denote them as $f(P,t)$ and $V(P,t)$ respectively. Both the fields are independent of the choice of the coordinate systems.

* Constant Vector function:- If $f_1, f_2, f_3$ are constant functions then $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ is called constant vector function.

* Scalar Point Function:- Let S be a domain in space. If to each point $P \in S$ there corresponds a scalar $\phi(P)$ then $\phi$ is called a scalar point function over the domain S.

Eg:- Let $\phi(P)$ be the density at any point P of a material body occupying a region $\mathbb{R}$. then $\phi$ is a scalar point function defined ~~over~~ for the region $\mathbb{R}$.

* Vector Point Function:- Let S be a domain in space. If to each point $P \in S$ there corresponds a vector $f(P)$, then f is called a vector point function over S.

Eg:- Let $f(P)$ denote the velocity of a particle at a point P in a region $\mathbb{R}$ then f is a vector point function defined in the region $\mathbb{R}$.

Level Sur... continu...

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**Level Surfaces:-** Let $f(x,y,z)$ be a single valued continuous scalar function defined at every point $P \in D$. Then $f(x,y,z)=c$, $c$ is constant, defines the equation of a surface and is called a "level surface" of the function.

For different values of $c$, we obtain different surfaces, no two of which intersect.

Eg:- (i) $x^2+y^2+z^2=c^2$ is level surfaces are spheres of radius $c$.

(ii) $x^2+y^2=\sqrt{[z-c]^2}$ is level surfaces are cones.

*** Differentiability of a vector function :-**

A vector function $\bar{v}(t)$ is said to be differentiable at a point, if the limit $\lim\limits_{\Delta t \to 0} \frac{\bar{V}(t+\Delta t)-\bar{V}(t)}{\Delta t}$ exists. If the limit exists, then we write is as $\bar{v}'(t)$ (or) $\frac{d\bar{v}}{dt}$.

if $\bar{v}_1(t)$, $\bar{v}_2(t)$, $v_3(t)$ are diffn fn's.

then $\bar{v}'(t) = v_1'(t)\,\hat{i} + v_2'(t)\,\hat{j} + v_3'(t)\,\hat{k}$.

*** Gradient of a Scalar field :-**

A vector diffn operator $\nabla$ is called 'del operator' and it is defined by

$$\nabla = \bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z} \quad (\text{or}) \quad \nabla = \Sigma i\frac{\partial}{\partial x}$$

Then the gradient of scalar field $f(x,y,z)$, is denoted by $\nabla f$ (or) grad $(f)$ is defined as

$$\nabla f = \text{grad } f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z} = \Sigma\, \bar{i}\frac{\partial f}{\partial x}$$

Eg:- (1) Find the gradient of the scalar fields

(i) $x^2y + xy^2 - z^2$ at $(3,1,1)$

Sol:- let $f(x,y,z) = x^2y + xy^2 - z^2$

$$\therefore \nabla f = i\frac{\partial}{\partial x}f + j\frac{\partial}{\partial y}f + k\frac{\partial}{\partial z}f$$

$$= i(2xy + y^2) + j(x^2 + 2xy) - k(2z)$$

at $(3,1,1)$, $\nabla f = 7i + 24j - 2k$

(2) If $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, $|\bar{r}| = r$ and $\hat{r} = \frac{\bar{r}}{r}$

then show that $\text{grad}\left(\frac{1}{r}\right) = -\frac{\hat{r}}{r^2}$

Sol: $\text{grad}\left(\frac{1}{r}\right) = i\frac{\partial}{\partial x}\frac{1}{r} + j\frac{\partial}{\partial y}\frac{1}{r} + k\frac{\partial}{\partial z}\frac{1}{r}$

$$= i\cdot -\frac{1}{r^2}\frac{\partial r}{\partial x} + j\cdot -\frac{1}{r^2}\frac{\partial r}{\partial y} + k\cdot -\frac{1}{r}\frac{\partial r}{\partial z}$$

$$= -\frac{1}{r^2}\left[i\frac{\partial r}{\partial x} + j\frac{\partial r}{\partial y} + k\frac{\partial r}{\partial z}\right]$$

$$= -\frac{1}{r^2}\left[i\frac{x}{r} + j\frac{y}{r} + k\frac{z}{r}\right] \quad \left(\because r = \sqrt{x^2+y^2+z^2}\right) \; (r^2 = x^2+y^2+z^2)$$

$$= -\frac{1}{r^2}\cdot\frac{1}{r}[xi + yj + zk]$$

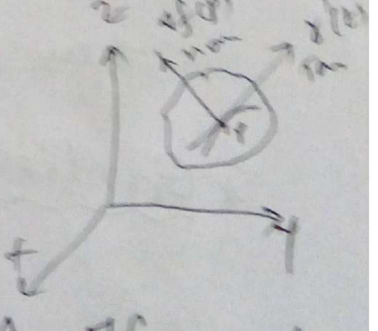
$$= -\frac{1}{r^2}\cdot\frac{1}{r}\bar{r}$$

$$= -\frac{\hat{r}}{r^2}$$

Properties:-

If $f$ & $g$ are s.p f^ns

1) $\nabla(f \pm g) = \nabla f \pm \nabla g$
2) $\nabla f = \bar{0} \Rightarrow f$ is s.p.f constant
3) $\nabla(fg) = f\nabla g + g\nabla f$
4) $\nabla(cf) = c\nabla f$
6)



UNIT VECTOR (NORMAL):-

The unit normal vector is $\hat{n} = \frac{\nabla f}{|\nabla f|}$ of $f$

① Find grad $f$ at the point $(1,1,-2)$ where

(i) $f = x^3 + y^3 + 3xyz$ (ii) $f = x^2y + y^2x + z^2$

③ Find $\nabla\phi$ if $\phi = x^2z^3 + yz^2$

*② Find the unit normal vector at any point $P(x,y,z)$ to the surface of a sphere with centre at the origin & radius as 1 unit.

Problems:-



1. Find a unit normal vector to the surface $xy^2+2yz=8$ at the point $(3,-2,1)$.

Sol: let $f(x,y,z)=xy^2+2yz-8$

$\nabla f = \cancel{i.y^2+j.2z+k.2y} = y^2 i+(2xy+2z)j+2yk$

unit normal vector $\hat{n}=\dfrac{\nabla f}{|\nabla f|}=\dfrac{y^2 i+(2z+2xy)j+2yk}{\sqrt{y^4+4y^2+(2z+2xy)^2}}$

at $(3,-2,1) \Rightarrow \hat{n}=\dfrac{4i-10j-4k}{\sqrt{33}}$.

[* The angle b/w two surfaces at a common point is the angle b/w their normals

2. Find the angle b/w the surfaces $x\log z=y^2-1$ and $x^2y=2-z$ at pt. at the point $(1,1,1)$

Sol: let $f_1 = x\log z-y^2+1$, $f_2=x^2y-2+z$ $\quad \cos\theta=\dfrac{|\nabla f_1 . \nabla f_2|}{|\nabla f_1|.|\nabla f_2|}$

$\nabla f_1(1,1,1) = -2j+k = n_1$

$\nabla f_2(1,1,1) = 2i+j+k = n_2$.

$\therefore \cos\theta=\dfrac{|n_1.n_2|}{|n_1||n_2|}=\dfrac{1}{\sqrt{30}} \Leftrightarrow \theta=\cos^{-1}\dfrac{1}{\sqrt{30}}$.

Note:- Properties

1) $\nabla(f+g)=\nabla f+\nabla g$

2) $\nabla(c_1f+c_2g)=c_1\nabla f+c_2\nabla g$

3) $\nabla(fg)=f\nabla g+g\nabla f$

4) $\nabla\left(\dfrac{f}{g}\right)=\dfrac{g\nabla f-f\nabla g}{g^2}$, $g\neq 0$.

5) $\nabla f=\bar{0} \Rightarrow f$ is constant spf

6) $\bar{r}=x\bar{i}+y\bar{j}+z\bar{k}$

$\Rightarrow d\bar{r}=(dx)\bar{i}+(dy)\bar{j}+(dz)\bar{k}$

7) If $\phi$ is spf

$d\phi=\dfrac{\partial\phi}{\partial x}dx+\dfrac{\partial\phi}{\partial y}dy+\dfrac{\partial\phi}{\partial z}dz$
$=\nabla\phi . d\bar{r}$

* Find angle b/w $x^2+y^2+z^2=9$ & $x^2+y^2-z=3$ at $(2,-1,2)$

* Find the angle of intersection of the spheres $x^2+y^2+z^2+4x-6y-8z=47$ at $x^2+y^2+z^2=29$ and $(+4,-3,2)$

* Find the angle b/w $xy^2z=3x+z^2$ & $3x^2-y^2+2z=1$ at $(1,-2,1)$ $\left(\cos^{-1}\dfrac{-3}{7\sqrt{6}}\right)$

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## * Directional Derivative:-

Let $f(P) = f(x,y,z)$ be a differentiable scalar field. Then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ denote rates of change of f in the directions of x, y, z axis respectively. If $f(x,y,z)=k$ is a level surface and $P_0$ is any point on it, then $\frac{\partial f}{\partial x}, \frac{\partial f}{\partial y}, \frac{\partial f}{\partial z}$ at $P_0(x_0, y_0, z_0)$ denote the slopes of the tangents lines in the direction which is call the directional derivative.

ie, $\frac{\partial}{\partial t} f(x(t), y(t), z(t))$ is the rate of change of f w.r.t the distance t.

We have $$\frac{\partial f}{\partial t} = \frac{\partial f}{\partial x}\frac{dx}{dt} + \frac{\partial f}{\partial y}\frac{dy}{dt} + \frac{\partial f}{\partial z}\frac{dz}{dt}$$

where $\frac{dx}{dt}, \frac{dy}{dt}, \frac{dz}{dt}$ are evaluated at $t=0$.

ie, at the point $P_0$, we write

$$\frac{\partial f}{\partial t} = \left(\bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}\right) \cdot \left(\bar{i}\frac{dx}{dt} + \bar{j}\frac{dy}{dt} + \bar{k}\frac{dz}{dt}\right)$$

$$= \Delta f \cdot \frac{d\bar{r}}{dt}, \quad (\text{when } \bar{r} = x\bar{i} + y\bar{j} + z\bar{k})$$

$\frac{d\bar{r}}{dt} = \hat{b}$ (a unit vector)

$\therefore$ The directional derivative of f in the direction of $\hat{b}$ is given by

Directional derivative $= \nabla f \cdot \hat{b}$

if u is direction vector $\hat{b} = \frac{u}{|u|}$.

* Problems:-

1) Find the directional derivative of $f(x,y,z) = xy^2 + 4xyz + z^2$ at the point $(1,2,3)$ in the direction of $3i+4j-5k$.

Sol: Given $f(x,y,z) = xy^2 + 4xyz + z^2$

$$\nabla f = i\frac{\partial f}{\partial x} + j\frac{\partial f}{\partial y} + k\frac{\partial f}{\partial z}$$

$$= i(y^2+4yz) + j(2xy+4xz) + k(4xy+2z)$$

and let $\bar{b} = 3i+4j-5k$.

unit vector of b is $\hat{b} = \frac{\bar{b}}{|\bar{b}|} = \frac{3i+4j-5k}{\sqrt{50}}$

at $(1,2,3)$, $\nabla f = 28i + 16j + 14k$.

$\therefore$ Directional derivative $D_b f(1,2,3) = \nabla f \cdot \hat{b}$

$$= (28i+16j+14k) \cdot \frac{3i+4j-5k}{5\sqrt{2}}$$

$$= \frac{78}{5\sqrt{2}}$$

*2) Find the directional derivative of $\phi(x,y,z) = x^2+y^2+z^2$ at the point $(1,-1,2)$ in the direction of vector $i+2j+3k$.

Sol:- let $\phi = x^2+y^2+z^2$

$$\nabla\phi = i2x + 2yj + 2zk$$

at $(1,-1,2) \Rightarrow \nabla\phi = 2i-2j+4k$.

and let $b = i+2j+3k$.

unit vector of b is $\hat{b} = \frac{b}{|b|} = \frac{i+2j+3k}{\sqrt{14}}$

$\therefore$ Directional vector $D_b\phi(1,-1,2) = \nabla\phi \cdot \hat{b}$

$$= (2i-2j+4k) \cdot \frac{i+2j+3k}{\sqrt{14}}$$

$$= \frac{10}{\sqrt{14}}$$

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Problems:

1) Find the directional derivative of $\phi = xyz$ at $(1,1,1)$ in the direction of the vector $i+j+k$.

2) Find the gradient and unit normal ~~ve~~ to the level surface $x^2+y^2-z=4$ at $(2,0,0)$.

*3) Determin the D.D of $f = xy^2+yz^3$ at $(2,-1,1)$ in direction of vector $i+2j+2k$.

* Maximum (or) max. value of directional derivative :-

The maximum directional derivative of the function $f(x,y,z)$ at the point $P_0(x_0,y_0,z_0)$ is define by $|\nabla f|$.

$$\left(\because \text{Max. } D_b f(x_0,y_0,z_0) = |\nabla f \cdot \hat{b}| = |\nabla f| \cdot |\hat{b}| = |\nabla f| \quad (\because |\hat{b}|=1)\right)$$

Problems:-

*1) Find the max. directional derivative of $f(x,y,z) = xy^2+yz^3$ at the point $(2,-1,1)$.

Sol: Max. D.D $= |\nabla f|$

$$\nabla f = i(y^2) + j(2xy+z^3) + k(3yz^2)$$

at $(2,-1,1)$, $\nabla f = i - 3j - 3k$

$$\therefore \text{Max. D.D} = \sqrt{1+9+9} = \sqrt{19}$$

(1) Find the max. D.D of (i) $\phi = x^2yz$ at $(1,4,1)$

(2) Find the greatest value of D.D of function $2x^2-y-z^4$ at $(2,-1,1)$

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(1) * Find the directional derivative of the function $\phi(x,y,z) = xy^2 + yz^2 + zx^2$ along the tangent to the curve $x = t,\ y = t^2,\ z = t^3$ at the point $(1,1,1)$

sol: Given $\phi = xy^2 + yz^2 + zx^2$

$$\nabla\phi = (y^2 + 2xz)i + (2xy + z^2)j + (2yz + x^2)k$$

$$\therefore \nabla\phi = 3i + 3j + 3k \quad , \text{ at } (1,1,1)$$

let $r$ be the position vector of any point the curve $x = t, y = t^2, z = t^3$ then $\bar{r} = t\bar{i} + t^2\bar{j} + t^3\bar{k}$

$$\therefore \frac{d\bar{r}}{dt} = \bar{i} + 2t\bar{j} + 3t^2\bar{k}$$

$$= \bar{i} + 2\bar{j} + 3\bar{k} \quad , \text{ at } (1,1,1)$$

$\because \frac{d\bar{r}}{dt}$ is the vector along the tangent to the curve & let $\bar{b} = i + 2j + 3k$

$$\hat{b} = \frac{\bar{b}}{|\bar{b}|} = \frac{i + 2j + 3k}{\sqrt{14}}$$

$$\therefore D.D = \nabla\phi \cdot \hat{b} = \frac{1}{\sqrt{14}}(3 + 6 + 9) = \frac{18}{\sqrt{14}}$$

(2) * Find the directional derivative of the fn $f = x^2 - y^2 + 2z^2$ at the point $P(1,2,3)$ in the direction of the line $PQ$ where $Q(5,0,4)$.

sol: Given $f = x^2 - y^2 + 2z^2$

$$\nabla f = 2xi - 2yj + 4zk$$

$$\therefore \nabla f = 2i - 4j + 12k \quad \text{at } P(1,2,3)$$

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let position vectors of $P$ & $Q$ are

$\bar{OP} = i+2j+3k, \quad \bar{OQ} = 5i+4k.$

let the vector $\bar{b} = \bar{PQ} = \bar{OQ} - \bar{OP}$

$= 4i-2j+k.$

$\therefore \hat{b} = \frac{1}{\sqrt{21}}(4i-2j+k)$

$\therefore D.D. = \nabla f \cdot \hat{b} = \frac{1}{\sqrt{21}}(4i-2j+k)\cdot(2i-4j+12k)$

$= \frac{1}{\sqrt{21}}(8+8+12)$

$= \frac{28}{\sqrt{21}}.$

(3) Find the unit tangent vector at any point on the curve $x = t^2+2,\ y = 4t-5,\ z = 2t^2-6t$, where $t$ is any variable.
*

Sol: The position vector of a point on the curve is $\bar{r}(t) = (t^2+2)\bar{i} + (4t-5)\bar{j} + (2t^2-6t)\bar{k}$

$\therefore$ tangent vector is $\bar{r}'(t) = 2t\,\bar{i} + 4\bar{j} + (4t-6)\bar{k}$

$\therefore$ Hence unite tangent vector $= \dfrac{2t\bar{i}+4\bar{j}+(4t-6)\bar{k}}{\sqrt{4t^2+16+(4t-6)^2}}$

at $t$ is any variable

(4) Find the ~~derict~~ directional derivative of $f(x,y) = x^2y^3+xy$ at $(2,1)$ in the direction of a unit vector which makes an angle of $\pi/3$ with axis
*

The unit vector $\theta = \pi/3$

Sol: $\Delta f = 5i+14j$, at $(2,1)$

$\hat{b} = i\cos\theta + j\sin\theta$

$= \frac{1}{2}i + \frac{\sqrt{3}}{2}j$

$\therefore D.D = \nabla f\cdot\hat{b} = \frac{5+14\sqrt{3}}{2}$

* Diverg...

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* Divergence of a vector field:

Let $\bar{f} = f_1(x,y,z)\bar{i} + f_2(x,y,z)\bar{j} + f_3(x,y,z)\bar{k}$ be a continuously differentiable vector function, then divergence of $\bar{f}$ denoted by "div f" (or) $\nabla . \bar{f}$ is defined as the Scalar field.

$$\text{div}\, \bar{f} = \nabla . \bar{f} = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) . (f_1\bar{i} + f_2\bar{j} + f_3\bar{k})$$

$$\therefore \nabla . \bar{f} = \frac{\partial f_1}{\partial x} + \frac{\partial f_2}{\partial y} + \frac{\partial f_3}{\partial z}$$

Note: (i) $\nabla . \bar{f} \neq \bar{f} . \nabla$ , $\nabla . \bar{f}$ is Scalar.

(ii) If $\bar{f}$ is a constant vector, then $\nabla . \bar{f} = 0$

(iii) $\nabla . f$ is no meaning, if $f$ is scalar

* Solenoidal Vector:-

If $\text{div}\, \bar{f} = 0$ then $\bar{f}$ is said to a Solenoidal vector point function.

* The Laplacian operator ($\nabla^2$):-

Let $\phi(x,y,z)$ be a s.p function.

$$\nabla . \nabla\phi = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) . \left(i\frac{\partial \phi}{\partial x} + j\frac{\partial \phi}{\partial y} + k\frac{\partial \phi}{\partial z}\right)$$

$$= \frac{\partial^2 \phi}{\partial x^2} + \frac{\partial^2 \phi}{\partial y^2} + \frac{\partial^2 \phi}{\partial z^2}$$

$$= \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right)\phi$$

$$= \nabla^2 \phi$$

$\therefore$ The operator $\nabla^2 = \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}$ is called the Laplacian operator.

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Problem :-

1) Find the divergence of the vector field
$f = (x^2y^2 - z^3)\,i + 2xyz\,j + e^{xyz}\,k$

Sol:
$$\nabla \cdot f = \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot\left((x^2y^2 - z^3)i + 2xyz\,j + e^{xyz}k\right)$$
$$= \frac{\partial}{\partial x}(x^2y^2 - z^3) + \frac{\partial}{\partial y}(2xyz) + \frac{\partial}{\partial z}e^{xyz}$$
$$= 2xy^2 + 2xz + xy\,e^{xyz}$$

* Curl of a vector field :-

let $\bar{f} = f_1(x,y,z)\,i + f_2(x,y,z)\,j + f_3(x,y,z)\,k$ be a continuously differentiable vector point function, then curl of $f$ is denoted by curl $\bar{f}$ (or) $\nabla\times\bar{f}$ and is defined as

$$\operatorname{curl} f = \nabla\times f = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ f_1 & f_2 & f_3 \end{vmatrix}$$

$$\text{(or)}\quad \nabla\times f = \sum\left[\frac{\partial f_3}{\partial y} - \frac{\partial f_2}{\partial z}\right]i$$

* Irrotational Vector :-

A vector point function $\bar{f}$ is said to be irrotational if curl $\bar{f} = 0$, i.e, $\nabla\times\bar{f} = 0$.

Note :- i) If $f$ is constant vector, curl $f = 0$.

ii) $\nabla\times f \neq f\times\nabla$

iii) $\nabla\times f = -(f\times\nabla)$

(iii) curl $(\bar{A} * \bar{B})$
$= \operatorname{curl}\bar{A} + \operatorname{curl}\bar{B}$

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Find the curl of the vector field

$$v = (x^2y^2 - z^3)\,i + 2xyz\,j + e^{xyz}\,k$$

Sol: let $v = v_1 i + v_2 j + v_3 k$.

$$\therefore \nabla \times v = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x^2y^2 - z^3 & 2xyz & e^{xyz} \end{vmatrix}$$

$$= i\left[xz e^{xyz} - 2xy\right] - j\left[yz e^{xyz} + 3z^2\right] + k\left[2yz - 2x^2y\right]$$

* Relations b/w $\nabla$, $\nabla\cdot$, $\nabla\times$ :-

(1) curl of gradient:

*

let $f$ be a differentiable scalar field. then

curl(grad $f$) = 0 (or) $\nabla \times (\nabla f) = 0$. , ~~where~~

~~$\nabla f$~~

Proof:- $\nabla f = \bar{i}\frac{\partial f}{\partial x} + \bar{j}\frac{\partial f}{\partial y} + \bar{k}\frac{\partial f}{\partial z}$

$$\therefore \nabla \times \nabla f = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \frac{\partial f}{\partial x} & \frac{\partial f}{\partial y} & \frac{\partial f}{\partial z} \end{vmatrix}$$

$$= i\left(\frac{\partial^2 f}{\partial y \partial z} - \frac{\partial^2 f}{\partial z \partial y}\right) - j\left[\frac{\partial^2 f}{\partial x \partial z} - \frac{\partial^2 f}{\partial z \partial x}\right] + k\left[\frac{\partial^2 f}{\partial x \partial y} - \frac{\partial^2 f}{\partial y \partial x}\right]$$

$$\therefore \nabla \times (\nabla f) = 0.$$

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(ii) * Divergence of curl:-

Let v be differentiable vector field. Then

$$\text{div}(\text{curl}\,\bar{v}) = 0 \quad (\text{or}) \quad \nabla\cdot(\nabla\times\bar{v}) = 0$$

Proof:- let $\bar{V} = v_1\bar{i} + v_2\bar{j} + v_3\bar{k}$

$$\nabla\times\bar{V} = i\left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] + j\left[\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right] + k\left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right]$$

$$\therefore \nabla\cdot(\nabla\times\bar{V}) = \frac{\partial}{\partial x}\left[\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right] + \frac{\partial}{\partial y}\left[\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right] + \frac{\partial}{\partial z}\left[\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right]$$

$$\therefore \nabla\cdot(\nabla\times\bar{v}) = 0$$

* Problems:-

1) If $\bar{f} = xy^2\,\bar{i} + 2x^2yz\,\bar{j} - 3yz^2\,\bar{k}$ find div f, curl f at the point (1,-1,1).

Sol:- $\text{div} f = \nabla\cdot f = \frac{\partial}{\partial x}xy^2 + \frac{\partial}{\partial y}2x^2yz + \frac{\partial}{\partial z}(-3yz^2)$

$$= y^2 + 2x^2z - 6yz$$

and 
$$\nabla\times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ xy^2 & 2x^2yz & -3yz^2 \end{vmatrix}$$

$$= i(-3z^2 - 2x^2y) - j(0-0) + k(4xyz - 2xy)$$

at (1,-1,1)

$$\nabla\cdot f = 1 + 2 + 6 = 9$$

and $\nabla\times f = -\bar{i} - 2\bar{k}$

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(2) Find div f and curl f where $f = \text{grad}(x^3+y^3+z^3-3xyz)$.

Sol: Given $f = \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right)(x^3+y^3+z^3-3xyz)$

$\therefore f = \bar{i}(3x^2-3yz)+\bar{j}(3y^2-3xz)+\bar{k}(3z^2-3xy)$

$\therefore \nabla \cdot f = \frac{\partial}{\partial x}(3x^2-3yz)+\frac{\partial}{\partial y}(3y^2-3xz)+\frac{\partial}{\partial z}(3z^2-3xy)$

$= 6x+6y+6z$

and $\nabla \times f = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x^2-3yz & 3y^2-3xz & 3z^2-3xy \end{vmatrix}$

$= i(-3x+3x) - j(-3y+3y) + k(-3z+3z)$

$= 0$

(3) If $\bar{r} = x\bar{i}+y\bar{j}+z\bar{k}$, find div $\bar{r}$. ($r$ is position vector)

Sol: div $\bar{r} = \nabla \cdot \bar{r} = \frac{\partial}{\partial x}x+\frac{\partial}{\partial y}y+\frac{\partial}{\partial z}z$

$= 1+1+1$

$= 3$

(4) Show that $3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k$ is solenoidal.

Sol: let $f = 3y^4z^2 i + 4x^3z^2 j - 3x^2y^2 k$

$\therefore \nabla \cdot f = \frac{\partial}{\partial x}3y^4z^2 + \frac{\partial}{\partial y}4x^3z^2 - \frac{\partial}{\partial z}3x^2y^2$

$= 0$

$\therefore f$ is solenoidal.

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⑤ Show that $\bar{f} = yz\,i + zx\,j + xy\,k$ is irrotational and find $\phi$ such that $\bar{f} = \nabla\phi$, where $\phi$ is scalar potential.

Sol:

Given $\bar{f} = yz\bar{i} + zx\bar{j} + xy\bar{k}$

$$\text{curl}\,\bar{f} = \nabla \times \bar{f} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ yz & zx & xy \end{vmatrix}$$

$$= i[x-x] - j[y-y] + k[z-z]$$

$$= 0$$

$\therefore \bar{f}$ is irrotational.

Now, Since $\bar{f} = \nabla\phi$.

$$yz\,i + zx\,j + xy\,k = i\cdot\frac{\partial\phi}{\partial x} + j\frac{\partial\phi}{\partial y} + k\frac{\partial\phi}{\partial z}$$

so that $\frac{\partial\phi}{\partial x} = yz$, $\frac{\partial\phi}{\partial y} = zx$, $\frac{\partial\phi}{\partial z} = xy$

$\therefore \partial\phi = yz\,\partial x$

$\therefore \phi = xyz + g(y,z)$

$\therefore \phi = xyz + k$, ($g$ is const fn)

$xz + \frac{\partial}{\partial y}g(y,z) = zx$

$g = g(z)$

$\frac{\partial}{\partial z}(xyz + g(z)) = xy$

$\frac{\partial g(z)}{\partial z} = 0$

$g = K$

$\phi = xyz + g(y,z)$

$\frac{\partial\phi}{\partial y} = xz + \frac{\partial g}{\partial y} = zx$

$g(y,z) = g(z)$

$\phi = xyz + g(z)$

$\frac{\partial\phi}{\partial z} = xy + \frac{\partial g}{\partial z} = xy$

$g = K$

⑥ If $\phi = x^2 - y^2$, show that $\nabla^2\phi = 0$.

Sol:-

$$\nabla^2\phi = \frac{\partial^2\phi}{\partial x^2} + \frac{\partial^2\phi}{\partial y^2} + \frac{\partial^2\phi}{\partial z^2}$$

$$= 2 - 2 + 0$$

$$= 0.$$

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7) If $\bar{r}$ is the position vector, then find $\text{curl}(\bar{r})$.

Sol: let $\bar{r} = xi + yj + zk$.

$$\text{curl } \bar{r} = \begin{vmatrix} i & j & k \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x & y & z \end{vmatrix}$$

$$= i(0-0) - j(0-0) + k(0-0)$$

$$= 0.$$

8) If $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$. show that $\text{div}\left(\frac{\bar{r}}{r^3}\right) = 0$.

Sol: Given $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$

$$= \sqrt{x^2+y^2+z^2}$$

$$\therefore r^2 = x^2+y^2+z^2$$

$$\therefore \text{div}\left(\frac{\bar{r}}{r^3}\right) = \nabla \cdot \frac{\bar{r}}{r^3}$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right) \cdot \left(\frac{x}{r^3}i + \frac{y}{r^3}j + \frac{z}{r^3}k\right)$$

$$= \frac{\partial}{\partial x}\frac{x}{r^3} + \frac{\partial}{\partial y}\frac{y}{r^3} + \frac{\partial}{\partial z}\frac{z}{r^3}$$

$$= \frac{1}{r^3} + x\left(\frac{-3}{r^4}\frac{\partial r}{\partial x}\right) + \frac{1}{r^3} - y\frac{3}{r^4}\frac{\partial r}{\partial y} + \frac{1}{r^3} - z\frac{3}{r^4}\frac{\partial r}{\partial z}$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left[x\frac{\partial r}{\partial x} + y\frac{\partial r}{\partial y} + z\frac{\partial r}{\partial z}\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^4}\left[\frac{x^2}{r} + \frac{y^2}{r} + \frac{z^2}{r}\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^5} \cdot r^2$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$$= 0.$$

$$= \sum \frac{\partial}{\partial x}\frac{x}{r^3}$$

$$= \sum \left\{\frac{1}{r^3} + x \cdot \frac{-3}{r^4} \cdot \frac{x}{r}\right.$$

$$= \frac{3}{r^3} - \frac{3}{r^5}\left[x^2+y^2+z^2\right]$$

$$= \frac{3}{r^3} - \frac{3}{r^3}$$

$$= 0$$

Properties:-

1) Show that $\operatorname{div}(A\times B) = B.\operatorname{curl}A - A.\operatorname{curl}B$

(or) $\nabla.(A\times B) = B.(\nabla\times A) - A.(\nabla\times B)$

Proof: $\operatorname{div}(\bar{A}\times B) = \nabla.(A\times B)$ $= \left(\bar{i}\frac{\partial}{\partial x}+\bar{j}\frac{\partial}{\partial y}+\bar{k}\frac{\partial}{\partial z}\right).(\bar{A}\times\bar{B})$

$= \sum i.\frac{\partial}{\partial x}(A\times B)$ $\left(\frac{\partial}{\partial x}\bar{a}\times\bar{b} = \bar{a}\times\frac{\partial\bar{b}}{\partial x} + \frac{\partial\bar{a}}{\partial x}\times\bar{b}\right)$

$= \sum i.\left(\frac{\partial A}{\partial x}\times B + A\times\frac{\partial B}{\partial x}\right)$

$= \sum i.\left(\frac{\partial A}{\partial x}\times B\right) + \sum i.\left(A\times\frac{\partial B}{\partial x}\right)$

$= \sum\left(i\times\frac{\partial A}{\partial x}\right).B + \sum i.-\left(\frac{\partial B}{\partial x}\times A\right)$

$= \sum\left(i\times\frac{\partial A}{\partial x}\right).B - \sum\left(i\times\frac{\partial B}{\partial x}\right).A$

$= (\nabla\times A).B - (\nabla\times B).A$

$= B.(\nabla\times A) - A.(\nabla\times B)$

$\left[\because \nabla\times A = i\times\frac{\partial A}{\partial x} + j\times\frac{\partial B}{\partial x} + k\times\frac{\partial B}{\partial z}\right.$

2) S.T $\nabla\times(\nabla\times A) = \nabla(\nabla.A) - \nabla^2 A$

Proof: $\nabla\times(\nabla\times A) = \sum i\times\frac{\partial}{\partial x}(\nabla\times A)$

Now $i\times\frac{\partial}{\partial x}(\nabla\times A) = i\times\frac{\partial}{\partial x}\left[i\times\frac{\partial A}{\partial x} + j\times\frac{\partial A}{\partial y} + k\times\frac{\partial A}{\partial z}\right]$

$= i\times\left[i\times\frac{\partial^2 A}{\partial x^2} + j\times\frac{\partial^2 A}{\partial x\partial y} + k\times\frac{\partial^2 A}{\partial x\partial z}\right]$

$= i\times\left(i\times\frac{\partial^2 A}{\partial x^2}\right) + i\times\left(j\times\frac{\partial^2 A}{\partial x\partial y}\right) + i\times\left(k\times\frac{\partial^2 A}{\partial x\partial z}\right)$

$= \left(i.\frac{\partial^2 A}{\partial x^2}\right)i - \frac{\partial^2 A}{\partial x^2}(i.i) + \left(i.\frac{\partial^2 A}{\partial x\partial y}\right)j - 0 + \left(i.\frac{\partial^2 A}{\partial x\partial z}\right)k - 0$

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$$= i\frac{\partial}{\partial x}\left(i\cdot\frac{\partial A}{\partial x}\right) + j\frac{\partial}{\partial y}\left(i\cdot\frac{\partial A}{\partial x}\right) + k\frac{\partial}{\partial z}\left(i\cdot\frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$= \nabla\left(i\cdot\frac{\partial A}{\partial x}\right) - \frac{\partial^2 A}{\partial x^2}$$

$$\therefore \sum i\times\frac{\partial}{\partial x}(\nabla\times A) = \nabla\sum\left(i\cdot\frac{\partial A}{\partial x}\right) - \Sigma\frac{\partial^2 A}{\partial x^2}$$

$$= \nabla(\nabla\cdot A) - \nabla^2 A$$

$$\therefore \nabla\times(\nabla\times A) = \nabla(\nabla\cdot A) - \nabla^2 A.$$

* If $\bar{a}$ is a constant vector and $\bar{r} = xi + yj + zk$ show that $\text{curl}(\bar{a}\times\bar{r}) = 2\bar{a}$.

Sol: let $\bar{a} = a_1 i + a_2 j + a_3 k$ is constant vector where $a_1, a_2, a_3$ are constants

& $\bar{r} = xi + yj + zk$

$$\therefore \text{curl}(\bar{a}\times\bar{r}) = \nabla\times(\bar{a}\times\bar{r})$$

$$= \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right)\times(\bar{a}\times\bar{r})$$

$$= i\times\left(\frac{\partial}{\partial x}(\bar{a}\times\bar{r})\right) + j\times\frac{\partial}{\partial y}(\bar{a}\times\bar{r}) + k\frac{\partial}{\partial z}(\bar{a}\times\bar{r})$$

$$= i\times\left(\bar{a}\times\frac{\partial\bar{r}}{\partial x}\right) + j\times\left(\bar{a}\times\frac{\partial\bar{r}}{\partial y}\right) + k\times\left(\bar{a}\times\frac{\partial\bar{r}}{\partial z}\right)$$

$$= i\times(\bar{a}\times i) + j\times(\bar{a}\times j) + k\times(\bar{a}\times k)$$

$$= (i\cdot i)\bar{a} - (i\cdot\bar{a})i + (j\cdot j)\bar{a} - (j\cdot\bar{a})j + (k\cdot k)\bar{a} - (k\cdot\bar{a})k$$

$$= \bar{a} - a_1 i + \bar{a} - a_2 j + \bar{a} - a_3 k$$

$$= 3\bar{a} - \bar{a}$$

$$= 2\bar{a}.$$

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Properties:-

1) curl $(fV) = (\text{grad} f) \times V + f \text{ curl} V$ (or)

$\nabla \times (fV) = \nabla f \times V + f(\nabla \times V)$, where $f$ is Scalar fn.

Proof:- $\text{curl}(fv) = \nabla \times (fv)$

$= \nabla \times (f v_1 i + f v_2 j + f v_3 k)$

$= \sum \left[ \frac{\partial}{\partial y} f v_3 - \frac{\partial}{\partial z} f v_2 \right] i$

$= \sum \left[ f \frac{\partial v_3}{\partial y} + v_3 \frac{\partial f}{\partial y} - \frac{\partial f}{\partial z} \cdot v_2 - f \frac{\partial v_2}{\partial z} \right] i$

$= \sum \left[ f \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) + v_3 \frac{\partial f}{\partial y} - v_2 \frac{\partial f}{\partial z} \right] i$

$= f \left[ \left( \frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z} \right) i + \left( \frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x} \right) j + \left( \frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y} \right) k \right]$

$+ \left( v_3 \frac{\partial f}{\partial y} - v_2 \frac{\partial f}{\partial z} \right) i + \left( v_1 \frac{\partial f}{\partial z} - v_3 \frac{\partial f}{\partial x} \right) j + \left( v_2 \frac{\partial f}{\partial x} - v_1 \frac{\partial f}{\partial y} \right) k$

$= f(\nabla \times V) + \left( i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \right) \times (v_1 i + v_2 j + v_3 k)$

$= f \text{ curl} V + \nabla f \times V$

$= f \text{ curl} V + (\text{grad} f) \times V$

II) $\text{div}(\text{grad} f) = \nabla^2 f$, where $f$ is scala fn, $\nabla$ is L.O

Proof: $\text{div}(\text{grad} f) = \nabla \cdot \nabla f$

$= \left( i \frac{\partial}{\partial x} + j \frac{\partial}{\partial y} + k \frac{\partial}{\partial z} \right) \cdot \left( i \frac{\partial f}{\partial x} + j \frac{\partial f}{\partial y} + k \frac{\partial f}{\partial z} \right)$

$= \frac{\partial^2 f}{\partial x^2} + \frac{\partial^2 f}{\partial y^2} + \frac{\partial^2 f}{\partial z^2}$

$= \nabla^2 f.$

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(iii) curl (curl v) = $\nabla(\nabla \cdot v) - \nabla^2 v$

Proof: curl (curl v) = $\nabla \times (\nabla \times v)$

$$= \left(\sum i \frac{\partial}{\partial x}\right) \times \left(\sum i \left(\frac{\partial v_3}{\partial y} - \frac{\partial v_2}{\partial z}\right)\right)$$

$$= \sum i \left[\frac{\partial}{\partial y}\left(\frac{\partial v_2}{\partial x} - \frac{\partial v_1}{\partial y}\right) - \frac{\partial}{\partial z}\left(\frac{\partial v_1}{\partial z} - \frac{\partial v_3}{\partial x}\right)\right]$$

$$= \sum i \left[\frac{\partial^2 v_2}{\partial y \partial x} + \frac{\partial^2 v_3}{\partial z \partial x} - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$= \sum i \left[\frac{\partial}{\partial x}\left(\frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) - \left(\frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$= \sum i \left[\frac{\partial}{\partial x}\left(\frac{\partial v_1}{\partial x} + \frac{\partial v_2}{\partial y} + \frac{\partial v_3}{\partial z}\right) - \left(\frac{\partial^2 v_1}{\partial x^2} + \frac{\partial^2 v_1}{\partial y^2} + \frac{\partial^2 v_1}{\partial z^2}\right)\right]$$

$$= \sum i \frac{\partial}{\partial x}(\nabla \cdot v) - \left(\frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2} + \frac{\partial^2}{\partial z^2}\right) \sum i v_1$$

$$= \nabla(\nabla \cdot v) - \nabla^2 v$$

i.e, curl (curl v) = grad (div. v) $- \nabla^2 v$

(1) If $\nabla \times \bar{A} = \bar{0}$, $\nabla \times \bar{B} = \bar{0}$, then find the value of Div $(\bar{A} \times \bar{B})$.

Sol: Div $(\bar{A} \times \bar{B}) = \nabla \cdot (\bar{A} \times \bar{B})$

$= \bar{B} \cdot \text{curl} \bar{A} - \bar{A} \cdot \text{curl} \bar{B}$

$= \bar{B} \cdot (\nabla \times \bar{A}) - \bar{A} \cdot (\nabla \times \bar{B})$

$= \bar{B} \cdot \bar{0} - \bar{A} \cdot \bar{0}$

$= 0$

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① Show that $\nabla^2 r^n = n(n+1) r^{n-2}$

$\underline{\underline{\text{Sol:}}}$ Let $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$

i.e, $r = \sqrt{x^2+y^2+z^2}$

(or) $r^2 = x^2+y^2+z^2$

$\therefore \nabla^2 r^n = \nabla \cdot \nabla r^n$

$= \nabla \cdot \left[ i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right] r^n$

$= \nabla \cdot i \sum \frac{\partial}{\partial x} r^n$

$= \nabla \cdot i \sum n r^{n-1} \frac{\partial r}{\partial x}$

$= \nabla \cdot i \sum n r^{n-1} \frac{1}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot \cancel{2}x$

$= \nabla \cdot i \sum n r^{n-2} x$

$= \nabla \cdot \left[ n r^{n-2} xi + n r^{n-2} yj + n r^{n-2} zk \right]$

$\cancel{\nabla \cdot n r^{n-2}}$

$= \left( i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) \cdot n r^{n-2} \left[ xi + yj + zk \right]$

$= \frac{\partial}{\partial x} n r^{n-2} x + \frac{\partial}{\partial y} n r^{n-2} y + \frac{\partial}{\partial z} n r^{n-2} z$

$= \sum \left[ n(n-2) r^{n-3} \cdot \frac{\partial r}{\partial x} \cdot x + n r^{n-2} \cdot 1 \right]$

$= \sum \left( n(n-2) r^{n-3} \cdot \frac{\cancel{2}x}{\cancel{2}\sqrt{x^2+y^2+z^2}} \cdot x + n r^{n-2} \right)$

$= \sum \left( n(n-2) r^{n-4} x^2 + n r^{n-2} \right)$

$= n(n-2) r^{n-4} (x^2+y^2+z^2) + 3n r^{n-2}$

$= n(n-2) r^{n-4} \cdot r^2 + 3n r^{n-2}$

$= (n^2 - 2n + 3n) r^{n-2}$

$= n(n+1) r^{n-2}$

$\nabla^2 r^n = \sum \frac{\partial^2}{\partial x^2} r^n$

$= \sum \frac{\partial}{\partial x} n r^{n-1} \cdot \frac{x}{r}$

$= \sum \frac{\partial}{\partial x} n r^{n-2} \cdot x$

$= n \sum \left[ (n-2) r^{n-3} \cdot \frac{x}{r} x + r^{n-2} \right]$

$= n \left[ \sum (n-2) r^{n-4} (x^2) + r^{n-2} \right]$

$= n \left[ (n-2) r^{n-4} (x^2+y^2+z^2) + 3 r^{n-2} \right]$

$= n \left[ (n-2) r^{n-2} + 3 r^{n-2} \right]$

$= n[n+1] r^{n-2}$

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(2) Show that $\nabla^2 r^n \bar{r} = n(n+3) r^{n-2} \bar{r}$, where $\bar{r} = xi + yj + zk$, and $r = |\bar{r}|$

Sol: Give $\bar{r} = xi + yj + zk$ and $r = |\bar{r}|$

$r = \sqrt{x^2+y^2+z^2}$

$\nabla^2 r^n \bar{r} = \nabla . \nabla r^n \bar{r}$

$= \nabla . \left( i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z} \right) r^n \bar{r}$

$\nabla^2 r^n \bar{r} = \Sigma \frac{\partial^2}{\partial x^2}(r^n \bar{r})$

$= \nabla . \left[ \Sigma i . \frac{\partial}{\partial x} r^n \bar{r} \right]$

$= \Sigma \frac{\partial}{\partial x}\left[ r^n . i + n r^{n-1} \frac{x}{r} \bar{r} \right]$

$= \nabla . \left[ \Sigma i . \left( n r^{n-1} \frac{1}{2\sqrt{x^2+y^2+z^2}} . 2x . \bar{r} + r^n i \right) \right]$

$= \nabla . \left[ \Sigma i \left( n r^{n-2} x \bar{r} + r^n i \right) \right]$

$= \nabla . \left( n r^{n-2} (xi + yj + zk) \bar{r} + 3 r^n \right]$

$= \nabla . \left( n r^{n-2} (\bar{r} . \bar{r}) + 3 r^n \right)$ $\quad (\bar{r} . \bar{r} = r^2)$

$= \nabla . \left( n r^{n-2} . r^2 + 3 r^n \right)$

$= \nabla . \left( n r^n + 3 r^n \right)$

$= \nabla . (n+3) r^n$

$= (n+3) \Sigma i . \frac{\partial}{\partial x} r^n$

$= (n+3) . \Sigma i . \left( n r^{n-1} \frac{1}{2\sqrt{x^2+y^2+z^2}} . 2x \right)$

$= (n+3) \Sigma i . n r^{n-2} x$

$= (n+3) . n r^{n-2} (xi + yj + zk)$

$= (n+3) n r^{n-2} . \bar{r}$

$= n(n+3) r^{n-2} \bar{r}$

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③ Show that $\nabla^2 \log r = \frac{1}{r^2}$

Sol: let $\bar{r} = xi + yj + zk$, so that $r = |\bar{r}| = \sqrt{x^2+y^2+z^2}$ *

$$\therefore \nabla^2 \log r = \nabla \cdot (\nabla \log r)$$

$$= \nabla \cdot \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\log r$$

$$= \nabla \cdot i \sum \frac{\partial}{\partial x}\log r$$

$$= \nabla \cdot i \sum \frac{1}{r}\cdot\frac{1}{2\sqrt{x^2+y^2+z^2}}(2x)$$

$$= \nabla \cdot i \sum \frac{x}{r^2}$$

$$= \nabla \cdot \frac{1}{r^2}[xi + yj + zk]$$

$$= \left(i\frac{\partial}{\partial x} + j\frac{\partial}{\partial y} + k\frac{\partial}{\partial z}\right)\cdot \frac{1}{r^2}(xi + yj + zk)$$

$$= i\sum \frac{\partial}{\partial x}\frac{1}{r^2}(xi+yj+zk)$$

$$= i\sum\left[\frac{1}{r^2}i + (xi+yj+zk)\cdot\frac{-2}{r^3}\cdot\frac{1}{2r}\cdot 2x\right]$$

$$= i\cdot\sum\left[\frac{1}{r^2}i - \frac{2x}{r^4}(xi+yj+zk)\right]$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(xi+yj+zk)\cdot(xi+yj+zk)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(x^2+y^2+z^2)$$

$$= \frac{3}{r^2} - \frac{2}{r^4}(r^2)$$

$$= \frac{1}{r^2}$$

$$\nabla^2 \log r = \Sigma \frac{\partial^2}{\partial x^2}\log r$$

$$= \Sigma \frac{\partial}{\partial x}\left(\frac{1}{r}\cdot\frac{x}{r}\right)$$

$$= \Sigma \frac{\partial}{\partial x}\frac{x}{r^2}$$

$$= \Sigma\left[\frac{r^2\cdot 1 - x\cdot 2r\cdot\frac{x}{r}}{r^4}\right]$$

$$= \Sigma\left[\frac{r^2 - 2x^2}{r^4}\right]$$

$$= \frac{1}{r^4}[3r^2 - 2r^2]$$

$$= \frac{1}{r^4}\cdot r^2$$

$$= \frac{1}{r^2}$$

Conserv

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## * Conservative vector field:-

A vector field $\bar{V}$ is said to be conservative if the vector function can be written as the gradient of a scalar function $f$.

i.e, $\bar{V} = \nabla f$

(i) $\bar{V}$ is c.v.f if $\text{curl}\,\bar{V} = \bar{0}$

(ii) A c.v.f is also irrotational, $\nabla \times \bar{V} = \bar{0}$

In a such a vector field, the work done in moving a particle from a point P to a point Q depends only on the points P and Q. and is independent of path along which the particle is displaced from P to Q. It may be noted that not every vector field is conservative.

**Problem:-**

1) Show that the vector field $\bar{f} = 2x(y^2+z^3)\bar{i} + 2x^2y\bar{j} + 3x^2z^2\bar{k}$ is conservative.

Sol: Given $\bar{f} = (2xy^2+2xz^3)\bar{i} + 2x^2y\,\bar{j} + 3x^2z^2\bar{k}$

let a scalar function $\phi(x,y,z)$.

So. $\bar{f} = \nabla\phi$.

$$2x(y^2+z^3)\bar{i} + 2x^2y\,\bar{j} + 3x^2z^2\bar{k} = \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)$$

$\therefore \dfrac{\partial\phi}{\partial x} = 2x(y^2+z^3)$, — ①  $\dfrac{\partial\phi}{\partial y} = 2x^2y$, — ②  $\dfrac{\partial\phi}{\partial z} = 3x^2z^2$ — ③

$$\int \partial\phi = \int 2x(y^2+z^3)\,\partial x$$

$\phi(x,y,z) = x^2y^2 + x^2z^3 + g(y,z)$ , $g(y,z)$ is constant to

Sub $\phi$ in eq ②.

$$\frac{\partial}{\partial y}\left[x^2y^2+x^2z^2+g(y,z)\right]=2x^2y$$

$$2x^2y+0+\frac{\partial g}{\partial y}=2x^2y$$

$$\frac{\partial g}{\partial y}=0$$

(or) $g=g(z)$ , $z$ is constant

From ③,

$$\frac{\partial}{\partial z}\left(x^2y^2+x^2z^3+g(z)\right)=3x^2z^2$$

$$0+3x^2z^2+\frac{\partial g}{\partial z}=3x^2z^2$$

$$\frac{\partial g}{\partial z}=0$$

$g=k$ ( $k$ is constant),

$\therefore$ Hence $\exists$ $\phi(x,y,z)=x^2y^2+x^2z^3+k$ is scalar fn. such that $\bar{f}=\nabla\phi$.

$\therefore$ the vector field $\bar{F}$ is Conservative.

② Show that the vector field defined by the vector function $\bar{V}=xyz(yz\bar{i}+xz\bar{j}+xy\bar{k})$ is Conservative.

$\left(f=\frac{1}{2}x^2y^2z^2+k\right)$

③ Find the constants $a,b,c$ so that the vector $\bar{F}=(x+2y+az)\bar{i}+(bx-3y-z)\bar{j}+(4x+cy+2z)\bar{k}$ is irrotational.

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Applications :-

* Velocity and Acceleration of a particle:-

Suppose that a body (or) a particle moves along a curve C. Then, the position vector of the particle (or) the body at time t is given by

$$\bar{r}(t) = x(t)\bar{i} + y(t)\bar{j} + z(t)\bar{k}$$

$$\text{Then } \bar{v}(t) = \bar{r}'(t) = x'(t)\bar{i} + y'(t)\bar{j} + z'(t)\bar{k}$$

$$\text{and } \bar{a}(t) = \bar{r}''(t) = x''(t)\bar{i} + y''(t)\bar{j} + z''(t)\bar{k}$$

are called the velocity and acceleration of the particle respectively

The scalar quantity $|\bar{v}(t)|$ is called the speed of the particle.

$$\text{i.e, } |\bar{v}(t)| = \sqrt{(x'(t))^2 + (y'(t))^2 + (z'(t))^2}$$

$$2x'(t) \cdot x''(t) + 2y'(t) \cdot y''(t) + 2z'(t) \cdot z''(t) = 6$$

Problems:-

1) The position vector of a moving particle is given by $\bar{r}(t) = t^3\bar{i} + t\bar{j} + t^2\bar{k}$, Determine the velocity, speed and acceleration of the particle in the direction of the motion.

Sol: Give $\bar{r}(t) = t^3\bar{i} + t\bar{j} + t^2 k$

$\therefore \bar{v}(t) = \bar{r}'(t) = 3t^2\bar{i} + \bar{j} + 2t\bar{k}$

speed $|\bar{v}(t)| = \sqrt{9t^4 + 1 + 4t^2}$

$\bar{a}(t) = \bar{r}''(t) = 6t\bar{i} + 2\bar{k}$

② If $\bar{r}(t) = (\cos t + \sin t)\bar{i} + (\sin t - \cos t)\bar{j} + t\bar{k}$.
Find velocity, speed, accelaration.

3) If a particle moves with constant speed c, then show that it's acceleration vector $\bar{a}(t)$ is $\perp^r$ to the velocity vector $\bar{v}(t)$.

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let $\bar{r}(t) = x\bar{i} + y\bar{j} + z\bar{k}$ be position vector

the the unit vector in the direction of the tangent is given by $\dfrac{\bar{r}'(t)}{|\bar{r}'(t)|}$.

(1) show that $\nabla[f(r)] = \dfrac{f'(r)}{r}\bar{r}$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$ is position vector of the $p(x,y,z)$.

Sol: Given $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$|\bar{r}| = \sqrt{x^2+y^2+z^2} = r$$

$$r^2 = x^2+y^2+z^2$$

$$\frac{\partial r}{\partial x} = \frac{x}{r}, \quad \frac{\partial r}{\partial y} = \frac{y}{r}, \quad \frac{\partial r}{\partial z} = \frac{z}{r}$$

$$\therefore \nabla[f(r)] = \left(\bar{i}\frac{\partial}{\partial x} + \bar{j}\frac{\partial}{\partial y} + \bar{k}\frac{\partial}{\partial z}\right) f(r)$$

$$= \sum \bar{i}\,\frac{\partial f(r)}{\partial x}$$

$$= \sum \bar{i}\, f'(r) \cdot \frac{\partial r}{\partial x}$$

$$= \sum \bar{i}\, f'(r) \cdot \frac{x}{r}$$

$$= \frac{f'(r)}{r}(x\bar{i} + y\bar{j} + z\bar{k})$$

$$= \frac{f'(r)}{r}\bar{r}$$

(2) If $a = x+y+z$, $b = x^2+y^2+z^2$, $c = xy+yz+zx$ then S.T $[\text{grad}\, a, \text{grad}\, b\ \text{grad}\, c] = 0$.

$$\begin{vmatrix} 1 & 1 & 1 \\ 2x & 2y & 2z \\ y+z & z+x & x+y \end{vmatrix}$$

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(3) S.T $\nabla(\log r) = \frac{\bar{r}}{r^2}$, where $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

Sol: let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k} \Rightarrow r = |\bar{r}|$

$r = \sqrt{x^2+y^2+z^2}$

$r^2 = x^2+y^2+z^2$

$$\therefore \nabla \log r = \Sigma \bar{i} \frac{\partial}{\partial x} \log r$$
$$= \Sigma \bar{i} \frac{1}{r} \cdot \frac{\partial r}{\partial x}$$
$$= \Sigma \bar{i} \frac{1}{r} \cdot \frac{x}{r}$$
$$= \frac{1}{r^2} \bar{r}.$$

(4) P.T $\nabla(r^n) = n r^{n-2} \bar{r}$

Sol: $$\nabla(r^n) = \Sigma \bar{i} \frac{\partial}{\partial x} r^n$$
$$= \Sigma \bar{i}\, n r^{n-1} \cdot \frac{x}{r}$$
$$= n r^{n-2} \bar{r}$$

(5) If $\bar{a}$ is constant vector, then S.T grad$(\bar{a}\cdot\bar{r}) = \bar{a}$

Sol: let $\bar{a} = a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$

$\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$$\bar{a}\cdot\bar{r} = a_1 x + a_2 y + a_3 z$$

$$\therefore \text{grad}(\bar{a}\cdot\bar{r}) = \Sigma \bar{i} \frac{\partial}{\partial x}(a_1 x + a_2 y + a_3 z)$$
$$= \Sigma \bar{i} a_1$$
$$= a_1\bar{i} + a_2\bar{j} + a_3\bar{k}$$
$$= \bar{a}.$$

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(5) S.T If $\bar{r} = x\hat{i} + y\hat{j} + z\hat{k}$. S.T $grad\left(\frac{1}{r}\right) = -\frac{\bar{r}}{r}$

Sol:-

$$\nabla \frac{1}{r} = \sum \hat{i} \frac{\partial}{\partial x} \frac{1}{r}$$

$$= \sum \frac{-1}{r^2} \cdot \frac{x\hat{i}}{r}$$

$$= \sum \frac{-x\hat{i}}{r^3} \qquad -\frac{1}{r^3}\bar{r}$$

$$= -$$

(6) Using vector notation prove that

$$\nabla^2 f(r) = f''(r) + \frac{2}{r} f'(r)$$

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