

UNIT-III

(1)

ORDINARY LINEAR DIFFERENTIAL EQ'S OF HIGHER ORDER

* O.L.D.E OF H.D WITH CONSTANT COEFFICIENTS:-

An equation of the form

$$\frac{d^m y}{dx^m} + P_1 \frac{d^{m-1} y}{dx^{m-1}} + P_2 \frac{d^{m-2} y}{dx^{m-2}} + \dots + P_m y = Q(x), \text{ where}$$

$P_1, P_2, \dots, P_m$ are real constants and $Q(x)$ is continuous function of x is called an ordinary L.D.Eqⁿ of order n with constant coefficients.

If $P_1, P_2, \dots, P_m$ are functions of x , then the above equation is called Ordinary L.D.Eqⁿ of order n with variable coefficients.

* SECOND ORDER L.D.Eq WITH CONSTANT COEFFICIENTS:-

$$\text{An equation of the form } \frac{d^2 y}{dx^2} + P \frac{dy}{dx} + Qy = R(x)$$

where P, Q are real constants and $R(x)$ is continuous function of x is called ordinary Second order L.D.Eqⁿ with constant coefficients.

OPERATOR FORM:-

Let us denote $\frac{d}{dx} = D, \frac{d^2}{dx^2} = D^2, \dots, \frac{d^n}{dx^n} = D^n$ are operators, Then the above eqⁿ can be written

$$\text{as } [D^m + P_1 D^{m-1} + P_2 D^{m-2} + \dots + P_m] y = Q(x)$$

$$\text{(or) } \boxed{f(D) y = Q(x)}$$

where $f(D) = D^m + P_1 D^{m-1} + \dots + P_m$ is a polynomial in D of degree n .

The symbol D stands for the operation of differentiation.

* GENERAL SOLUTION OF $f(D)y = Q(x)$:-

Given $(D^n + P_1 D^{n-1} + \dots + P_n)y = Q(x)$, $D \equiv \frac{d}{dx}$ ①

Let $y_1, y_2, \dots, y_n$ be n independent solutions of ①, then $y = C_1 y_1 + C_2 y_2 + \dots + C_n y_n$ is the most general solution of ①, and it is called 'Complementary function (C.F.)'. it is denoted by y_c (or) y_c .

$$\therefore y_c = C_1 y_1 + C_2 y_2 + \dots + C_n y_n.$$

of $y = v$ (or) $y = y_p$ is any particular solution of eq ①. then y_p is called "Particular Integral" of eq ①.

$\therefore$ The complete General solution of ①

is $y = C.F + P.I$

i.e., $y = y_c + y_p$

* If $Q(x) = 0$, then there is no P.I of eq ①.

* C.F is general solution of $f(D)y = 0$.

* The equation $f(D)y = 0$ is Homogeneous Eqn.

* Then equation $f(D)y = Q$, ($Q(x) \neq 0$) is non-Homogeneous equation.

(2)

* TO FIND THE GENERAL SOLⁿ (or) C.F OF $f(D)y=0$:

A algebraic equation $f(m)=0$ [Replace D by m in $f(D)$]. i.e $m^n + P_1 m^{n-1} + \dots + P_n = 0$, where $P_1, P_2, \dots, P_n$ are real constants is called auxilliary eqⁿ of $f(D)y=0$.

Since the A-Eqⁿ $f(m)=0$ is a polynomial eqⁿ of degree n , it will have n roots say that $m_1, m_2, \dots, m_n$.

They are may be (i) real and distinct
(ii) real and some equal
(iii) real and complex.
(iv) real and irrational.

case (i)
* If " α " be a real root of $f(m)=0$, then C.F $e^{\alpha x}$ where C is arbitrary constant, is the corresponding part of the C.F.

If $m_1, m_2, \dots, m_n$ are all real and distinct then then the Solution (or) C.F is

$$y = y_c = C_1 e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}.$$

case (ii) If $m_1, m_2, \dots, m_n$ are real and some equal roots, then i.e.

If $m_1 = m_2 = m_3 = \dots = m_n$, then

$$\text{Solution } y = y_c = (C_1 + C_2 x) e^{m_1 x} + C_2 e^{m_2 x} + \dots + C_n e^{m_n x}$$

If $m_1 = m_2 = m_3 = \dots = m_n$ then

$$y = y_c = (C_1 + C_2 x + C_3 x^2 + \dots + C_n x^{n-1}) e^{m_1 x}$$

case (iii) If $a+ib$ is complex root of $f(m)=0$ then $a-ib$ is also root of $f(m)=0$.

Then the ~~C.F~~ part of C.F is $e^{ax} [C_1 \cos bx + C_2 \sin bx]$.

If $a \pm ib, m_3, m_4, \dots, m_n$ are roots of $f(m) = 0$

Then solution is

$$y = y_c = e^{ax} [c_1 \cos bx + c_2 \sin bx] + c_3 e^{m_3 x} + \dots + c_n e^{m_n x}$$

Case (iv):- If $a \pm ib, a \pm ib, m_5, \dots, m_n$ are roots of $f(m) = 0$.

then solution

$$y = y_c = e^{ax} [(c_1 + c_2 x) \cos bx + (c_3 + c_4 x) \sin bx] + c_5 e^{m_5 x} + \dots + c_n e^{m_n x}$$

Case (v):- If $a \pm \sqrt{b}$ is irrational roots of A.E. a^m .
Then C.F. $y_c = e^{ax} [c_1 \cosh \sqrt{b} x + c_2 \sinh \sqrt{b} x]$.

* PROBLEMS:-

1) Solve $\frac{d^2 y}{dx^2} - a^2 y = 0, a \neq 0$

Sol:

Given operator form $(D^2 - a^2)y = 0, D = \frac{d}{dx}$
 $f(D)y = 0$

where $f(D) = D^2 - a^2$

$\therefore$ A.E. $f(m) = 0$

$$m^2 - a^2 = 0$$

$$m = \pm a$$

Gen. sol is $y = c_1 e^{ax} + c_2 e^{-ax}$,

where c_1, c_2 are arbitrary const.

2) Find the complementary function for $(D^3 - 3D^2 + 3D - 1)y = 0$

Sol:

Given $(D^3 - 3D^2 + 3D - 1)y = 0, D = \frac{d}{dx}$
 $f(D)y = 0$

A.E. $f(m) = m^3 - 3m^2 + 3m - 1 = 0$

$$m = 1,$$

$$\therefore m = 1, 0 \pm i\sqrt{3}$$

$$\begin{array}{r|rrrr} 1 & 1 & -3 & 3 & -1 \\ & & 1 & 0 & 3 \\ \hline & 1 & 0 & 3 & 0 \end{array}$$

$$m^2 + 3 = 0$$

$$y = y_c = c_1 e^x + e^{0x} [c_2 \cos \sqrt{3}x + c_3 \sin \sqrt{3}x] \quad m = \pm i\sqrt{3}$$

3) which use complementary function when the A.Eqn have the roots $2+3i$ and $3-2i$.

Sol:

Given $2+3i$ and $3-2i$ are roots of A.Eqn.
then it's conjugates $2-3i$, $3+2i$ are also roots of A.Eqn.

Hence the roots are $2\pm 3i$, $3\pm 2i$

$$\therefore \text{C.F } y_c = e^{2x} [c_1 \cos 3x + c_2 \sin 3x] + e^{3x} [c_3 \cos 2x + c_4 \sin 2x]$$

4) Solve $\frac{d^3 y}{dx^3} - 3 \frac{dy}{dx} + 2y = 0$.

Sol:

Given $(D^3 - 3D + 2)y = 0$.

$$f(D)y = 0$$

A.Eqn $f(m) = m^3 - 3m + 2 = 0$

$$m = 1, 1, -2$$

A.S. $y = (c_1 + c_2 x)e^x + c_3 e^{-2x}$

5) Solve $(D^4 - 2D^3 - 3D^2 + 4D + 4)y = 0$.

roots $m = -1, -1, 2, 2$.

$$\therefore y = (c_1 + c_2 x)e^{-x} + (c_3 + c_4 x)e^{2x}$$

6) Solve $\frac{d^2 y}{dx^2} + \frac{dy}{dx} + y = 0$

roots $m = -\frac{1}{2} \pm i\frac{\sqrt{3}}{2}$

$$\therefore y = e^{-\frac{1}{2}x} \left[c_1 \cos \frac{\sqrt{3}}{2}x + c_2 \sin \frac{\sqrt{3}}{2}x \right]$$

7) Solve $(D^3 - 14D + 8)y = 0$

roots $m = -4, \pm\sqrt{2}$

$$y_c = c_1 e^{-4x} + e^{\sqrt{2}x} [c_2 \cosh \sqrt{2}x + c_3 \sinh \sqrt{2}x]$$

8) Find the G. solⁿ of $y'' + y' - 2y = 0$, $y(0) = 4, y'(0) = 1$

Sol:

A-Eqⁿ $f(m) = m^2 + m - 2 = 0$

$m = 1, -2$

$y = c_1 e^x + c_2 e^{-2x}$

at $y(0) = 4$

$y(0) = c_1 + c_2 = 4$

$y'(0) = 1 \Rightarrow y'(x) = c_1 e^x - 2c_2 e^{-2x}$
 $y'(0) = c_1 - 2c_2 = 1$

$\therefore c_1 + c_2 = 4$

$c_1 - 2c_2 = 1$

$\begin{array}{r} - + \\ \hline \end{array}$

$3c_2 = 3$

$c_2 = 1$

$\therefore c_1 = 3$

$\therefore$ Required S.S $y = 3e^x + e^{-2x}$

H.W.P :-

1) what is the C.F of $\frac{d^3 y}{dx^3} + 4 \frac{dy}{dx} = 0$.

2) what is solution of $(D^2 - 1)y = 0$

3) solve $(D^2 - 4D + 4)y = 0$

4) solve $(D^4 - 1)y = 0$

5) solve $(D^2 + 2D - 3)y = 0$, $y(0) = 0, y'(0) = 4$

6) solve $(D^3 + 1)y = 0$

7) solve $(D^2 + 1)^2 y = 0$

* INVERSE (OR) INTEGRAL OPERATOR:-

(4)

If D is differential operator Then D^{-1} (or) $\frac{1}{D}$ is called Inverse (or) integral operator of D .

* If Q is a fn of x , Then $\frac{1}{D} Q$ is integral of Q .

Ex: $\frac{1}{D} \cos 3x = \frac{\sin 3x}{3}$

* If $f(D)$ is the differential operation polynomial in D and Q is fn of x . Then

$$\frac{1}{f(D)} Q(x) \text{ is particular value (or)}$$

also called Particular Integral, containing no arbitrary constants.

* If Q is any fn of x and 'a' is a constant, then a particular value of $\frac{1}{D-a} Q$ is equal to

$$e^{ax} \int e^{-ax} Q dx$$

$$\text{i.e. } \frac{1}{D-a} Q = e^{ax} \int e^{-ax} Q dx$$

$$\text{also } \frac{1}{D+a} Q = e^{-ax} \int e^{ax} Q dx$$

Note:-

$$\frac{1}{(D-a)(D-b)} Q = \frac{1}{D-a} \left[\frac{1}{D-b} Q \right]$$

$$= e^{ax} \int e^{-ax} \left[e^{bx} \int e^{-bx} Q dx \right] dx$$

* PROBLEMS:-

1) Find $\frac{1}{D} x^2$, $\frac{1}{D^3} \cos x$

Sol: $\frac{1}{D} x^2 = \int x^2 dx = \frac{x^3}{3}$

$$\frac{1}{D^3} \cos x = \iiint \cos x dx = -\sin x.$$

2) Find Particular value of $\frac{1}{D+1} x$.

Sol: $\frac{1}{D+1} x = e^{-x} \int e^x \cdot x dx$
 $= e^{-x} [x e^x - e^x]$
 $= x - 1.$

3) Find Particular value of $\frac{1}{(D-2)(D-3)} e^{2x}$.

Sol: $\frac{1}{(D-2)(D-3)} e^{2x} = \frac{1}{D-2} \left[e^{3x} \int e^{-3x} \cdot e^{2x} dx \right]$
 $= \frac{1}{D-2} [e^{3x} (-e^{-x})]$
 $= e^{2x} \int e^{-2x} \cdot (-e^{2x}) dx$
 $= -x e^{2x}$



LINEAR D.EQ:-

A L.D. Eqⁿ is in which the dependent variable and its derivatives occur only in the first degree and are not multiplied together.

* GENE

* GENERAL SOLUTION OF $f(D)y = Q(x)$:-

We know that $y = y_p$ is a particular solution of $f(D)y = Q$ containing no arbitrary constants and $y = y_c$ is g. soln of $f(D)y = 0$, then $y = y_c + y_p$ is the general solution of $f(D)y = Q$.

If $y = y_c$ is a g. soln of $f(D)y = 0$ then y_c is the C.F of $f(D)y = Q$. and the particular Integral is defined by $y_p = \frac{1}{f(D)} Q$.

Now we will discuss methods to find P.I of $f(D)y = Q(x)$.

* GENERAL METHOD TO FIND P.I:-

Given $f(D)y = Q(x)$

when $\frac{1}{f(D)}$ is expressed as partial fractions

let $f(D) = (D - \alpha_1)(D - \alpha_2) \dots (D - \alpha_n)$

$$\therefore \text{P.I } y_p = \frac{1}{f(D)} Q = \frac{1}{(D - \alpha_1)(D - \alpha_2) \dots (D - \alpha_n)} Q$$

$$= \left[\frac{A_1}{D - \alpha_1} + \frac{A_2}{D - \alpha_2} + \dots + \frac{A_n}{D - \alpha_n} \right] Q$$

$$= A_1 e^{\alpha_1 x} \int e^{-\alpha_1 x} Q dx + A_2 e^{\alpha_2 x} \int e^{-\alpha_2 x} Q dx + \dots + A_n e^{\alpha_n x} \int e^{-\alpha_n x} Q dx$$

Sol:

1) Solve $(D^2 - 4D - 5)y = x e^{-x}$
given $f(D)y = Q(x)$
where $f(D) = D^2 - 4D - 5$, $Q(x) = x e^{-x}$

$$\therefore \text{G. soln } y = y_c + y_p$$

$$\text{Take H.Eqn } f(m) = 0 \\ m^2 - 4m - 5 = 0 \\ (m+1)(m-5) = 0$$

$$m = -1, 5$$

$$\therefore y_c = c_1 e^{-x} + c_2 e^{5x}$$

$$\text{Now P.I. } y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{D^2 - 4D - 5} x e^{-x}$$

$$= \frac{1}{(D+1)(D-5)} x e^{-x}$$

$$= \frac{1}{6} \left[\frac{-1}{D+1} + \frac{1}{D-5} \right] x e^{-x}$$

$$= \frac{1}{6} \left[e^{5x} \int e^{-5x} x e^{-x} dx - e^{-x} \int e^x x e^{-x} dx \right]$$

$$= \frac{1}{6} \left[e^{5x} \int x e^{-6x} dx - e^{-x} \int x dx \right]$$

$$= \frac{1}{6} \left[e^{5x} \left[-\frac{x e^{-6x}}{6} - \frac{e^{-6x}}{36} \right] - e^{-x} \frac{x^2}{2} \right]$$

$$= \frac{1}{6} \left[e^{-x} \left[-\frac{x}{6} - \frac{1}{36} \right] - e^{-x} \frac{x^2}{2} \right]$$

$$y_p = \frac{1}{6} e^{-x} \left[-\frac{x}{6} - \frac{1}{36} - \frac{x^2}{2} \right]$$

$$\therefore \text{G. soln } y = c_1 e^{-x} + c_2 e^{5x} - \frac{1}{6} e^{-x} \left[\frac{x}{6} + \frac{1}{36} + \frac{x^2}{2} \right]$$

2) Solve $(D^2+1)y = \tan x$

Sol: Given $f(D) = D^2+1$, $Q(x) = \tan x$

A-Eqn $m^2+1=0 \Rightarrow m = \pm i = 0 \pm i$

$\therefore y_c = c_1 \cos x + c_2 \sin x$

Now P.I. $y_p = \frac{1}{f(D)} Q$

$= \frac{1}{D^2+1} \tan x$

$= \frac{1}{(D-i)(D+i)} \tan x$

$= \frac{1}{2i} \left(\frac{1}{D-i} - \frac{1}{D+i} \right) \tan x$

$= \frac{1}{2i} \left[\frac{1}{D-i} \tan x - \frac{1}{D+i} \tan x \right] \text{ --- (1)}$

Now $\frac{1}{D-i} \tan x = e^{ix} \int e^{-ix} \tan x dx$

$= e^{ix} \int (\cos x - i \sin x) \frac{\sin x}{\cos x} dx$

$= e^{ix} \int \left(\sin x - i \frac{1-\cos^2 x}{\cos x} \right) dx$

$= e^{ix} \left[-\cos x - i \left(\int (\sec x - \cos x) dx \right) \right]$

$= e^{ix} \left[-\cos x - i \log(\sec x + \tan x) + i \sin x \right]$

$= e^{ix} \left[-(\cos x - i \sin x) - i \log(\sec x + \tan x) \right]$

$\frac{1}{D-i} \tan x = e^{ix} \left[-e^{-ix} - i \log(\sec x + \tan x) \right]$

Replace i by $-i$

$\frac{1}{D+i} \tan x = e^{-ix} \left[-e^{ix} + i \log(\sec x + \tan x) \right]$

$\therefore$ from (1) $y_p = \frac{1}{2i} \left[-1 - i e^{ix} \log(\sec x + \tan x) + 1 - i e^{-ix} \log(\sec x + \tan x) \right]$

$$= \frac{1}{2i} \left[-i (e^{ix} + e^{-ix}) (\log(\sec x + \tan x)) \right]$$

$$y_p = -\cos x \cdot \log(\sec x + \tan x)$$

$$\therefore \text{G.S. } y = y_c + y_p$$

$$\therefore y = c_1 \cos x + c_2 \sin x - \cos x \cdot \log(\sec x + \tan x)$$

* H.W.P:-

1) Solve $(D^2 - 5D + 6)y = x^4 e^{4x}$

$$2, 3, y_p = \frac{1}{4} e^{4x} (2x^2 - 3)$$

2) Solve $(D^2 + a^2)y = \sec ax$

$$\frac{2}{a} \sin ax + \frac{1}{a^2} \cos ax \log(\cos ax)$$

3) Solve $(D^2 + 4D + 3)y = e^{e^x}$

$$-3, -1, y_p = e^{e^x} (e^{-2x} - e^{-3x})$$

4) Solve $y'' + y = \csc x$

* TO FIND P.I OF $f(D)y = Q(x)$ IN SOME SPECIAL CASES:

* (1) P.I OF $f(D)y = Q(x)$, when $Q = e^{ax}$, a is constant:-

$$\text{Given } f(D)y = Q(x), Q = e^{ax}$$

$$\text{P.I } y_p = \frac{1}{f(D)} Q(x)$$

$$= \frac{1}{f(D)} e^{ax}$$

Replacing D by a .

$$\boxed{\therefore y_p = \frac{1}{f(a)} e^{ax}}, \text{ where } f(a) \neq 0$$

if $f(a) = 0$, Then $(D - a)$ is factor of $f(D)$.

if a' is a root repeated k times for $f(a) = 0$

$$\text{Then } y_p = \frac{x^k}{k!} \frac{1}{\phi(a)} e^{ax}, \quad \phi(a) \neq 0.$$

* PROBLEMS:-

1) Solve $\frac{d^2 y}{dx^2} + 4 \frac{dy}{dx} + 3y = e^{2x}$ (or) P.S of $f(x) = e^{2x}$

Sol:

Given $(D^2 + 4D + 3)y = e^{2x}$

$$f(D)y = Q(x)$$

$$f(D) = D^2 + 4D + 3, \quad Q(x) = e^{2x}$$

A.E.qⁿ $m^2 + 4m + 3 = 0$

$$m = -1, -3$$

$$\therefore y_c = C_1 e^{-x} + C_2 e^{-3x}$$

$$\text{and } y_p = \frac{1}{f(D)} Q(x) = \frac{1}{(D^2 + 4D + 3)} e^{2x}$$

$$= \frac{1}{(D+1)(D+3)} e^{2x}$$

Replac D by 2

$$= \frac{1}{15} e^{2x}$$

$\therefore$ G.S $y = y_c + y_p$

$$\therefore y = C_1 e^{-x} + C_2 e^{-3x} + \frac{1}{15} e^{2x}$$

2) Solve $(D^2 + 4D + 13)y = 2e^{-x}$

Given $f(D) = D^2 + 4D + 13, \quad Q = 2e^{-x}$

Sol:

A.E.qⁿ $m^2 + 4m + 13 = 0 \Rightarrow m = -1 \pm i\sqrt{3}$

$$\therefore y_c = e^{-x} (C_2 \cos 3x + C_3 \sin 3x)$$

Now $y_p = \frac{1}{D^2 + 4D + 13} \cdot 2e^{-x} \quad (a = -1)$

$$= 2 \cdot \frac{1}{(-1)^2 + 4(-1) + 13}$$

$$= \frac{1}{5} e^{-x}$$

$$\therefore y = e^{-x} (C_2 \cos 3x + C_3 \sin 3x) + \frac{1}{5} e^{-x}$$

3) Find the P.I of $y'' - 3y' + 2y = e^{3x}$.

Sol:

Given $(D^2 - 3D + 2)y = e^{3x}$
 $f(D)y = Q(x)$

$$\therefore \text{P.I } y_p = \frac{1}{D^2 - 3D + 2} e^{3x}$$

$$= \frac{1}{(D-1)(D-2)} e^{3x} \quad (a=3)$$

$$\therefore y_p = \frac{1}{2} e^{3x}$$

4) Solve $(D^3 - 5D^2 + 8D - 4)y = e^{2x}$

Sol:

Given $f(D)y = Q$

A.E.M $m^3 - 5m^2 + 8m - 4 = 0$

$$(m-1)(m-2)^2 = 0$$

$$m = 1, 2, 2$$

$$= y_c = c_1 e^x + (c_2 + c_3 x) e^{2x}$$

Now $y_p = \frac{1}{(D-1)(D-2)^2} e^{2x} \quad (a=2)$

$$= \frac{1}{2-1} \cdot \frac{x^2}{2!} e^{2x} \quad (2 \text{ is repeated Two times})$$

$$= \frac{x^2}{2} e^{2x}$$

$$\therefore y = c_1 e^x + (c_2 + c_3 x) e^{2x} + \frac{x^2}{2} e^{2x}$$

5) Solve $(D^3 + 1)y = 3 + 5e^x$

Sol:

$f(D) = D^3 + 1, \quad Q(x) = 3 + 5e^x$

A.E.M $m^3 + 1 = 0$

$$m = -1, \frac{1 \pm i\sqrt{3}}{2}$$

$$y_c = e^{-x} + e^{\frac{1}{2}x} \left[c_2 \cos \frac{\sqrt{3}}{2}x + c_3 \sin \frac{\sqrt{3}}{2}x \right]$$

$$\begin{aligned}
 y_p &= \frac{1}{f(D)} Q(x) \\
 &= \frac{1}{D^3+1} (5e^x + 3) \\
 &= 5 \frac{1}{D^3+1} e^x + 3 \frac{1}{D^3+1} e^{0x} \\
 &\quad (a=1) \quad (a=0)
 \end{aligned}$$

$$y_p = \frac{5}{2} e^x + 3$$

$$\therefore \text{Ans } y = y_c + y_p$$

H.W.P:-

1) Solve $(D^2 + 6D + 9)y = 2e^{-3x}$ $2e^{-3x}$

2) Solve $y'' - y' - 2y = 3e^{2x}$, $y(0)=0, y'(0)=-2$ $e^{2x} + e^{-x} + 2$

3) Solve $(D^2 + 2D + 1)y = e^{-x}$ $\frac{x^2}{2} e^{-x}$

4) Solve $(D^3 - 6D^2 + 11D - 6)y = e^{-2x} + e^{-3x}$ $\frac{-1}{120} (2e^{-2x} + e^{-3x})$

5) Solve $(D^2 - 3D + 2)y = \cosh x$ $-\frac{1}{2} x e^x + \frac{1}{12} e^{-x}$

6) Find P.I of $(D^2 + 5D + 6)y = e^x$

*7) Solve $(D^3 + 4D)y = 1$, *8) solve $(D^2 - 5D + 6)y = e^{4x}$

II
* P.I of $f(D)y = Q(x)$, when $Q(x) = \sin bx$ (or) $\cos bx$, b is constant:-

Given $f(D)y = Q(x)$

P.I. $y_p = \frac{1}{f(D)} Q(x)$

$$= \frac{1}{f(D)} \sin bx \text{ (or) } \cos bx$$

let $f(D) = \phi(D^2)$

$$= \frac{1}{\phi(D^2)} \sin bx \text{ (or) } \cos bx$$

Replace D^2 by $-b^2$

$$\therefore y_p = \frac{1}{\phi(-b^2)} \sin bx \text{ (or) } \cos bx, \phi(-b^2) \neq 0$$

$$\text{if } \phi(-b^2) = 0.$$

$$\text{i.e., } \frac{1}{D^2+b^2} \sin bx = \frac{-x}{2b} \cos bx$$

$$\frac{1}{D^2+b^2} \cos bx = \frac{x}{2b} \sin bx$$

* PROBLEMS:-

1) Solve $(D^2+3D+2)y = \sin 3x$

Sol:

Given $f(D)y = Q(x)$

A.E. $m^2+3m+2=0 \Rightarrow m = -1, -2$

$$y_c = C_1 e^{-x} + C_2 e^{-2x}$$

$$y_p = \frac{1}{f(D)} Q(x) = \frac{1}{D^2+3D+2} \sin 3x$$

(b=3)
Reple D^2 by -3^2

$$= \frac{1}{-3^2+3D+2} \sin 3x$$

$$= \frac{1}{3D-7} \sin 3x$$

$$= \frac{1}{3D-7} \times \frac{3D+7}{3D+7} \sin 3x$$

$$= \frac{3D+7}{9D^2-49} \sin 3x$$

(b=3)

$$= \frac{(3D+7) \sin 3x}{9(-9^2)-49}$$

$$= \frac{1}{-130} [3.3 \cos 3x + 7 \sin 3x]$$

$$y_p = -\frac{1}{130} [9 \cos 3x + 7 \sin 3x]$$

∴ soln is $y = y_c + y_p$

Q) Solve $(D^2 - 4D + 3)y = \cos 2x$

Sol:

$$f(D)y = Q(x)$$

A. Eqn $m^2 - 4m + 3 = 0 \Rightarrow m = 1, 3$

$$\therefore y_c = c_1 e^x + c_2 e^{3x}$$

$$\begin{aligned} y_p &= \frac{1}{D^2 - 4D + 3} \cos 2x \quad (b=2) \\ &= \frac{1}{-2^2 - 4D + 3} \cos 2x = \frac{1}{-4D - 1} \cos 2x \\ &= -\frac{1}{4D+1} \times \frac{4D-1}{4D-1} \cos 2x \\ &= -\frac{(4D-1)}{(16D^2-1)} \cos 2x \\ &= \frac{(-4D+1) \cos 2x}{16(-2^2)-1} \\ &= \frac{-1}{-64-1} [4 \cdot -2 \sin 2x - \cos 2x] \\ y_p &= \frac{-1}{65} [8 \sin 2x + \cos 2x] \\ \therefore \text{S.S } y &= y_c + y_p \end{aligned}$$

3) Find the P.I of $(D^2 + 16)y = \sin 4x$
Sol:

$$f(D) = D^2 + 16, \quad Q(x) = \sin 4x$$

$$\begin{aligned} \text{P.I} \Rightarrow y_p &= \frac{1}{f(D)} Q(x) \\ &= \frac{1}{D^2 + 16} \sin 4x \quad (b=4) \\ &= \frac{-1}{2(4)} \cos 2x \\ &= -\frac{1}{8} \cos 2x. \end{aligned}$$

* 4) Solve $y'' + y = \sin x + \cosh x + 2$

Sol:

Given $(D^2 + 1)y = \sin x + \cosh x + 2$

A.Eqn $m^2 + 1 = 0 \Rightarrow m = \pm i$

$y_c = C_1 \cos x + C_2 \sin x$

$y_p = \frac{1}{D^2 + 1} (\sin x + \cosh x + 2)$

$= \frac{1}{D^2 + 1} \sin x + \frac{1}{D^2 + 1} \cosh x + \frac{1}{D^2 + 1} 2$

$= \frac{-1}{2(1)} \cos x + \frac{1}{D^2 + 1} \left(\frac{e^x + e^{-x}}{2} \right) + \frac{1}{D^2 + 1} 2 e^{0x}$ (a=0)

$= -\frac{1}{2} \cos x + \frac{1}{2} \left[\frac{1}{D^2 + 1} e^x + \frac{1}{D^2 + 1} e^{-x} \right] + 2 \cdot \frac{1}{0^2 + 1}$

$= -\frac{1}{2} \cos x + \frac{1}{2} \left[\frac{1}{2} e^x + \frac{1}{2} e^{-x} \right] + 2$

$y_p = -\frac{1}{2} \cos x + \frac{1}{2} \cosh x + 2$

$\therefore$ Ans $y = y_c + y_p$

5) Solve $\frac{d^3 y}{dx^3} + 4 \frac{dy}{dx} = \sin 2x$

Sol:

Given $(D^3 + 4D)y = \sin 2x$

A.Eqn $m^3 + 4m = 0 \Rightarrow m = 0, \pm 2i$

$\therefore y_c = \cancel{C_1 x} + \cancel{C_2 x} + C_3 e^{-4x}$
 $= C_1 + C_2 \cos 2x + C_3 \sin 2x$

$\therefore y_p = \frac{1}{D^3 + 4D} \sin 2x$

$= \frac{1}{(D^2 + 4)D} \sin 2x = \frac{1}{D^2 + 4} \left(\frac{1}{D} \sin 2x \right)$

$= \frac{1}{D^2 + 4} \cdot \frac{-\cos 2x}{2} = -\frac{1}{2} \frac{1}{D^2 + 4} \cos 2x$

$= -\frac{1}{2} \cdot \frac{1}{4} \sin 2x = -\frac{1}{8} \sin 2x$

$\therefore y = y_c + y_p$

H.W.P:-

- 1) Find P.I of $(D^2+9)y = \cos 3x$. and $\frac{1}{D^2-1} \sin 3x$
- 2) Find C.F of $(D^2+9)y = \sin 3x$. $\frac{1}{D^2-1} \sin 3x$
- 3) Solve $y''+4y'+4y = 4\cos x + 3\sin x$ $\sin x$
- 4) Solve $(D^2-4)y = 2\cos^2 x$. $-\frac{1}{4} - \frac{\cos 2x}{8}$
- 5) Solve $(D^2+D+1)y = \sin 2x$. $-\frac{1}{13} (2\cos 2x + 3\sin 2x)$
- *6) Solve $(D^3+4D)y = \cos 3t$, $D \equiv \frac{d}{dt}$
- 7) ~~For same D~~ solve $y''+y = \sin x + \cos x + 2$

* III. P.I of $f(D)y = Q(x)$, when $Q(x) = x^k$, $k \in \mathbb{Z}^+$:-

Given $f(D)y = Q(x)$. $Q(x) = x^k$, $k \in \mathbb{N}$

P.I $y_p = \frac{1}{f(D)} x^k$

Here, reduce the $f(D)$ to the form $(1 \pm \phi(D))$ by taking out the lowest degree term from $f(D)$.

$$= \frac{1}{1 \pm \phi(D)} x^k$$

$\therefore y_p = [1 \pm \phi(D)]^{-1} x^k$, expand by Binomial theorem

Here using expansions

$$(1+x)^{-1} = 1 - x + x^2 - \dots$$

$$(1-x)^{-1} = 1 + x + x^2 + \dots$$

$$(1+x)^{-2} = 1 - 2x + 3x^2 - \dots$$

$$(1-x)^{-2} = 1 + 2x + 3x^2 + \dots$$

NOTE:- $D^{n+1} x^n = D^{n+2} x^n = \dots = 0$.

$$D^n x^n = n!$$

* PROBLEMS:-

1) Solve $(D^2 + D + 1)y = x^3$

Sol:

Given $f(D) = D^2 + D + 1$, $Q(x) = x^3$

A-Eqn $m^2 + m + 1 = 0$

$$m = \frac{-1 \pm i\sqrt{3}}{2}$$

$$\therefore y_c = e^{-x/2} \left[C_1 \cos \frac{\sqrt{3}}{2} x + C_2 \sin \frac{\sqrt{3}}{2} x \right]$$

$$y_p = \frac{1}{f(D)} x^3 = \frac{1}{D^2 + D + 1} x^3$$

$$= [1 + (D^2 + D)]^{-1} x^3$$

$$= [1 + (D^2 + D) + (D^2 + D)^2 - (D^2 + D)^3 + \dots] x^3$$

$$= [1 - D^2 - D + D^4 + D^2 + 2D^3 - D^6 - D^3 - 3D^4 - 3D^5 + \dots] x^3$$

$$= [1 - D + D^3] x^3, \quad (\because D^4, D^5, \dots \text{neglecting})$$

$$y_p = x^3 - 3x^2 + 6x$$

$$\therefore y = y_c + y_p$$

2) Solve $(D^3 - 3D - 2)y = x^2$

Sol:

Given $m^3 - 3m - 2 = 0$

$$m = 2, -1, -1$$

$$y_c = (C_1 + C_2 x) e^{-x} + C_3 e^{2x}$$

$$y_p = \frac{1}{D^3 - 3D - 2} x^2$$

$$= \frac{1}{-2 \left[1 + \frac{3D - D^3}{2} \right]} x^2$$

$$= -\frac{1}{2} \left[1 + \frac{3D - D^3}{2} \right]^{-1} x^2$$

$$\begin{aligned}
&= -\frac{1}{2} \left[1 - \left(\frac{3D-D^3}{2} \right) + \left(\frac{3D-D^3}{2} \right)^2 - \left(\frac{3D-D^3}{2} \right)^3 + \dots \right] x^2 \quad (11) \\
&= -\frac{1}{2} \left[1 - \frac{3D}{2} + \frac{D^3}{2} + \frac{9D^2}{4} + \frac{D^6}{4} - \frac{3D^4}{2} + \dots \right] x^2 \\
&= -\frac{1}{2} \left[1 + \frac{9}{4} D^2 - \frac{3D}{2} \right] x^2 \\
&= -\frac{1}{2} \left[x^2 + \frac{9}{4} \cdot 2x - \frac{3}{2} \cdot 2x \right] \\
\therefore y_p &= -\frac{1}{2} \left[x^2 - 3x + \frac{9}{2} \right]
\end{aligned}$$

∴ soln. $y = y_c + y_p$

3) Solve $(D^2 - 4D + 4)y = e^{2x} + x + \sin 3x$

Sol:

Given $f(D) = D^2 - 4D + 4$
 A.E.M $f(m) = m^2 - 4m + 4 = 0$
 $(m-2)^2 = 0$
 $m = 2, 2$

$$y_c = (C_1 + C_2 x) e^{2x}$$

Now $y_p = \frac{1}{(D^2 - 4D + 4)} (e^{2x} + x + \sin 3x)$

$$\begin{aligned}
&= \frac{1}{(D-2)^2} e^{2x} + \frac{1}{(D-2)^2} x + \frac{1}{D^2 - 4D + 4} \sin 3x \quad (b=3) \\
&= \frac{x^2}{2!} e^{2x} + \frac{1}{(-2)^2 \left[1 - \frac{D}{2} \right]^{-2}} x + \frac{1}{-3^2 - 4D + 4} \sin 3x \\
&= \frac{x^2}{2} e^{2x} + \frac{1}{4} \left(1 - \frac{D}{2} \right)^{-2} x + \frac{1}{-4D - 5} \sin 3x \\
&= \frac{x^2}{2} e^{2x} + \frac{1}{4} \left[1 + 2 \cdot \frac{D}{2} + 3 \frac{D^2}{4} + \dots \right] x - \frac{1}{4D + 5} \times \frac{4D - 5}{4D - 5} \sin 3x \\
&= \frac{x^2}{2} e^{2x} + \frac{1}{4} [x + Dx] - \frac{(4D - 5)}{16D^2 - 25} \sin 3x \\
&= \frac{x^2}{2} e^{2x} + \frac{1}{4} (x + 1) - \frac{1}{16(-3^2) - 25} (12 \cos 3x - 5 \sin 3x) \\
y_p &= \frac{x^2}{2} e^{2x} + \frac{1}{4} (x + 1) + \frac{1}{169} (12 \cos 3x - 5 \sin 3x) \\
\therefore \text{Soln } y &= y_c + y_p
\end{aligned}$$

* H.W.P. :-

1) Solve $(D^2 - 3D + 2)y = 2x^2$

2) Solve $(D^3 - 3D^2 - 10D + 24)y = x + 3$

3) Solve $(D^2 - 4D + 4)y = 8x^2 + e^{2x}$

* 4) Solve $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x + \sin 2x$

~~(5) Solve $(D^3 + D^2 + 2D + 2)y = \cos 4x \cos x$~~

~~Solve $y''' + 2y'' - y' + 2y = 1 - 4x^2$~~

* IV P.I of $f(D)y = Q(x)$, When $Q(x) = e^{ax} V$, where a is a constant, V is a function of x :-

Given $f(D)y = Q(x)$, when $Q = e^{ax} V$.

P.I. $y_p = \frac{1}{f(D)} Q(x)$

$= \frac{1}{f(D)} e^{ax} V$

Replace D by $D+a$

$= e^{ax} \frac{1}{f(D+a)} V$

Now Here Proceeding, $\frac{1}{f(D+a)}$ with

V alone by the previous methods.

* PROBLEMS:-

1) Solve $(D^2 + 2)y = e^x \cos x$

Given $f(D)y = Q(x)$

A.E.M $m^2 + 2 = 0 \Rightarrow m = \pm \sqrt{2}i$

$\therefore y_c = C_1 \cos \sqrt{2}x + C_2 \sin \sqrt{2}x$

Sol:-

Now $y_p = \frac{1}{f(D)} Q$

(12)

$$= \frac{1}{D^2+2} e^x \cos x$$

(a=1)

Replace D by D+1

$$= e^x \frac{1}{(D+1)^2+2} \cos x$$

$$= e^x \frac{1}{D^2+2D+3} \cos x$$

(b=1)

$$= e^x \frac{1}{-1+2D+3} \cos x$$

$$= e^x \frac{1}{2(D+1)} \cos x$$

$$= \frac{e^x}{2} \frac{1}{D+1} \times \frac{D-1}{D-1} \cos x$$

$$= \frac{e^x}{2} \frac{D-1}{D^2-1} \cos x$$

$$= \frac{e^x}{2} \left(-\frac{1}{2}\right) (-\sin x - \cos x)$$

$$y_p = \frac{e^x}{4} (\sin x + \cos x)$$

$\therefore$ G. soln $y = y_c + y_p$

2) Solve $(D^2+4D+3)y = e^{-x} \sin x + x$

Given $f(D)y = Q$

A.E. Eqn $f(m) = 0 \Rightarrow m^2+4m+3 = 0$
 $(m+1)(m+3) = 0$
 $m = -1, -3$

$$\therefore y_c = c_1 e^{-x} + c_2 e^{-3x}$$

Now $y_p = \frac{1}{D^2+4D+3} (e^{-x} \sin x + x)$

$$= \frac{1}{D^2+4D+3} e^{-x} \sin x + \frac{1}{D^2+4D+3} x$$

(a=-1)

$$= e^{-x} \frac{1}{(D-1)^2+4(D-1)+3} \sin x + \frac{1}{3 \left[1 + \frac{D^2+4D}{3}\right]} x$$

$$\begin{aligned}
&= e^{-x} \frac{1}{D^2+2D} \sin x + \frac{1}{3} \left[1 + \left(\frac{D^2+4D}{3} \right)^{-1} \right] x \\
&= e^{-x} \frac{1}{-1^2+2D} \sin x + \frac{1}{3} \left[1 - \frac{D^2+4D}{3} + \left(\frac{D^2+4D}{3} \right)^2 - \dots \right] x \\
&= e^{-x} \frac{1}{2D-1} \times \frac{2D+1}{2D+1} \sin x + \frac{1}{3} \left[1 - \frac{4}{3} D \right] x \\
&= e^{-x} \frac{2D+1}{4D^2-1} \sin x + \frac{1}{3} \left[x - \frac{4}{3} \right] \\
&= e^{-x} \frac{2D+1}{4(-1^2)-1} \sin x + \frac{1}{9} [3x-4] \\
\therefore y_p &= -\frac{e^{-x}}{5} [2\cos x + \sin x] + \frac{1}{9} (3x-4)
\end{aligned}$$

G. soln $y = y_c + y_p.$

f 3)
sol:

Solve $(D^3 + D^2 + 2D + 2)y = \cosh x \cdot \cos 2x$

Given $f(D)y = Q(x)$

A.E.M $m^3 + m^2 + 2m + 2 = 0$

$m = -1, \pm \sqrt{2}i$

$y_c = c_1 e^{-x} + c_2 \cos \sqrt{2}x + c_3 \sin \sqrt{2}x.$

Now $y_p = \frac{1}{D^3 + D^2 + 2D + 2} \cosh x \cdot \cos 2x$
 $= \frac{1}{D^3 + D^2 + 2D + 2} \left(\frac{e^x + e^{-x}}{2} \right) \cos 2x$

$= \frac{1}{2} \left[\frac{1}{D^3 + D^2 + 2D + 2} e^x \cos 2x + \frac{1}{D^3 + D^2 + 2D + 2} e^{-x} \cos 2x \right]$
 $(a=1)$

$= \frac{1}{2} \left[\frac{e^x}{(D+1)^3 + (D+1)^2 + 2(D+1) + 2} \cos 2x + e^{-x} \frac{1}{(D-1)^3 + (D-1)^2 + 2(D-1) + 2} \cos 2x \right]$

$= \frac{1}{2} \left[e^x \frac{1}{D^3 + 4D^2 + 7D + 6} \cos 2x + e^{-x} \frac{1}{D^3 - 2D^2 + D + 2} \cos 2x \right]$

$= \frac{1}{2} \left[e^x \frac{1}{D(-2) + 4(-2)^2 + 7D + 6} \cos 2x + e^{-x} \frac{1}{D(-2) - 2(-2)^2 + D + 2} \cos 2x \right]$

$= \frac{1}{2} \left[e^x \frac{1}{-4D - 16 + 7D + 6} \cos 2x + e^{-x} \frac{1}{-4D + 8 + D + 2} \cos 2x \right]$

$= \frac{1}{2} \left[e^x \frac{1}{3D - 10} \cos 2x + e^{-x} \frac{1}{-3D + 10} \cos 2x \right]$

$$= \frac{1}{2} \left[e^{\lambda} \frac{3D+10}{9D^2-100} \cos 2\lambda + \bar{e}^{\lambda} \cdot \frac{-(3D+10)}{9D^2-100} \cos 2\lambda \right] \quad (13)$$

$$= \frac{1}{2} \left[e^{\lambda} \frac{-68m2\lambda + 10 \cos 2\lambda}{9(-2^2) - 100} - \bar{e}^{\lambda} \frac{-68m2\lambda + 10 \cos 2\lambda}{9(-2^2) - 100} \right]$$

$$= \frac{1}{2} \left[e^{\lambda} \frac{68m2\lambda + 10 \cos 2\lambda}{136} + \bar{e}^{\lambda} \frac{68m2\lambda + 10 \cos 2\lambda}{136} \right]$$

$$= \frac{1}{2} [e^{\lambda} - \bar{e}^{\lambda}] \left[\frac{1}{136} (68m2\lambda - 10 \cos 2\lambda) \right]$$

$$y_p = \frac{1}{68} \sin 2\lambda \cdot (3 \sin 2\lambda - 5 \cos 2\lambda)$$

$$\therefore \text{G.Sol}^n \quad y = y_c + y_p$$

4) Solve $(D^2 + 2D - 3)y = x^2 e^{-3x}$

Sol:

Given $f(D)y = Q$.

A.E. eqⁿ $m^2 + 2m - 3 = 0 \Rightarrow (m-1)(m+3) = 0$
 $m = 1, -3$

$$y_c = C_1 e^{\lambda} + C_2 e^{-3\lambda}$$

Now $y_p = \frac{1}{D^2 + 2D - 3} e^{-3\lambda} x^2$ (a = -3)

$$= e^{-3\lambda} \frac{1}{(D-3)^2 + 2(D-3) - 3} x^2$$

$$= e^{-3\lambda} \frac{1}{D^2 - 4D} x^2$$

$$= e^{-3\lambda} \frac{1}{-4D \left[1 - \frac{D}{4}\right]} x^2$$

$$= -e^{-3\lambda} \frac{1}{4D} \left[1 - \frac{D}{4}\right]^{-1} x^2$$

$$= -e^{-3\lambda} \frac{1}{4D} \left[1 + \frac{D}{4} + \frac{D^2}{16} + \frac{D^3}{64} + \dots\right] x^2$$

$$= -\frac{1}{4} e^{-3\lambda} \left[\frac{1}{D} + \frac{1}{4} + \frac{D}{16} + \frac{D^2}{64}\right] x^2$$

$$= -\frac{1}{4} e^{-3\lambda} \left[\frac{x^3}{3} + \frac{1}{4} x^2 + \frac{1}{8} x + \frac{1}{32}\right]$$

$$y_p = -\frac{1}{4} e^{-3\lambda} \left[\frac{x^3}{3} + \frac{1}{4} x^2 + \frac{1}{8} x\right] \quad (\because \frac{1}{128} e^{-3\lambda} \text{ deleted})$$

$$\therefore \text{G.Sol}^n \quad y = y_c + y_p$$

H.W.P:-

- 1) Solve $(D^2 - 5D + 6)y = e^x \sin x$
- * 2) Solve $(D^3 - 4D^2 - D + 4)y = e^{3x} \cos 2x$
- 3) Solve $(D^2 + 1)y = x^2 e^{3x}$
- 4) Solve $(D^2 - 4D + 4)y = e^{2x} + \cos 2x + e^x \sin 2x$
- * 5) Solve $(D^2 - 4D - 5)y = x e^x$
- * 6) Solve $(D^3 + 2D^2 + D)y = e^{2x} + x^2 + x + \sin 2x$
- * 6) $(D^2 + 1)y = \sin x \sin 2x + e^x x^2$
- * 7) $(D^3 - 3D + 2)y = \sinh x \cdot \cos 2x$

* V P.I of $f(D)y = Q(x)$, when $Q(x) = x^m V$, where $m \in \mathbb{N}$,
 V is a function of x :-

Given $f(D)y = Q(x)$, $Q(x) = x^m V$.

We have $e^{iax} = \cos ax + i \sin ax$ ($\because V$ is $\sin bx$ (or) $\cos bx$)
 $(a$ is constant)

$\cos ax = \text{R.P of } e^{iax}$

$\sin ax = \text{I.P of } e^{iax}$

$\therefore y_p = \frac{1}{f(D)} x^m V \sin bx$ (or) $y_p = \frac{1}{f(D)} x^m \cos bx$

$= \frac{1}{f(D)} (\text{I.P of } e^{ibx}) x^m$ (or) $y_p = \frac{1}{f(D)} (\text{R.P of } e^{ibx}) x^m$

$= \text{I.P of } e^{ibx} \frac{1}{f(D+ib)} x^m$ (or) $y_p = \text{R.P of } e^{ibx} \frac{1}{f(D+ib)} x^m$

Here, Proceeding the previous method - III

$$\frac{e^x}{10} (\sin x + 3 \cos x)$$

$$-\frac{e^{3x}}{20} (\sin 2x + 7 \cos 2x)$$

$$\frac{e^{2x}}{25} (25x^2 - 30x + 13)$$

$$\frac{x^2}{2} e^{2x} - \frac{1}{6} \sin 2x +$$

$$\frac{e^x}{25} (4 \cos x - 3 \sin x)$$

$$-\frac{1}{6} e^x \left(\frac{x^2}{2} + \frac{x}{2} \right)$$

PROBLEMS:-

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1) Solve $(D^2+4)y = x \cos x$.

Sol:

Given $f(D)y = Q(x)$

A. Eqn $m^2+4=0 \Rightarrow m = \pm 2i$

$\therefore y_c = c_1 \cos 2x + c_2 \sin 2x$

Now $y_p = \frac{1}{f(D)} Q$

$= \frac{1}{D^2+4} x \cos x$

$(e^{ix} = \cos x + i \sin x)$

$= \frac{1}{D^2+4} x \cdot (R.P.O.f. e^{ix})$

$= R.P.O.f. \frac{1}{D^2+4} e^{ix} \cdot x$

$(a=i)$

$= R.P.O.f. e^{ix} \frac{1}{(D+i)^2+4} x$

$= R.P.O.f. e^{ix} \frac{1}{D^2+2iD+3} x$

$= R.P.O.f. e^{ix} \frac{1}{3} \left[\frac{1}{1 + \frac{D^2+2iD}{3}} \right] x$

$= R.P.O.f. e^{ix} \cdot \frac{1}{3} \left[1 + \left(\frac{D^2+2iD}{3} \right) \right] x$

$= \frac{1}{3} R.P.O.f. e^{ix} \left[1 - \frac{D^2+2iD}{3} \right] x$

$= \frac{1}{3} R.P.O.f. e^{ix} \left[x - \frac{2}{3}i \right]$

$= \frac{1}{3} R.P.O.f. (\cos x + i \sin x) \left(x - \frac{2}{3}i \right)$

$y_p = \frac{1}{3} \left[x \cos x + \frac{2}{3} \sin x \right]$

$\therefore$ Gr. soln $y = y_c + y_p$

* 2) Solve $(D^2 - 1)y = x \sin x + x^2 e^x$

Sol:

Given $f(D)y = Q$

A.E.M $m^2 - 1 = 0 \Rightarrow m = \pm 1$
 $\therefore y_c = C_1 e^x + C_2 e^{-x}$

Now $y_p = \frac{1}{D^2 - 1} (x \sin x + x^2 e^x)$
 $= \frac{1}{D^2 - 1} x \sin x + \frac{1}{D^2 - 1} e^x x^2$ (a=1)
 $= \frac{1}{D^2 - 1} x (\text{I.p.f of } e^{ix}) + e^x \frac{1}{(D+1)^2 - 1} x^2$ (a=i)
 $= \text{I.p.f of } e^{ix} \frac{1}{(D+i)^2 - 1} x + e^x \frac{1}{D^2 + 2D} x^2$
 $= \text{I.p.f of } e^{ix} \frac{1}{D^2 + 2iD - 2} x + e^x \frac{1}{2D [1 + \frac{D}{2}]} x^2$
 $= \text{I.p.f of } e^{ix} \frac{1}{-2 [1 - (\frac{D^2 + 2iD}{2})]} x + e^x \cdot \frac{1}{2D} [1 + \frac{D}{2}]^{-1} x^2$
 $= \text{I.p.f of } e^{ix} \cdot (-\frac{1}{2}) [1 - \frac{D^2 + 2iD}{2}]^{-1} x + \frac{e^x}{2D} [1 - \frac{D}{2} + \frac{D^2}{4}] x^2$
 $= \text{I.p.f of } e^{ix} \cdot (-\frac{1}{2}) [1 + iD] x + \frac{e^x}{2} \cdot \frac{1}{D} [x^2 x + \frac{1}{2}]$
 $= \text{I.p.f of } e^{ix} \cdot (-\frac{1}{2}) [x + i] + \frac{e^x}{2} [\frac{1}{2} x - \frac{x^2 + 1}{2}]$
 $= -\frac{1}{2} \text{I.p.f} [\cos x + i \sin x] [x + i] + \frac{e^x}{2} [3x - x^2] [\frac{x^3}{3} - \frac{x^2}{2} + \frac{x}{2}]$
 $= -\frac{1}{2} [\cos x + x \sin x] + \frac{x e^x}{12} (3x - x^2) [2x^2 - 3x + 3]$
 $\therefore \text{G. Sol } y = y_c + y_p$

3) Solve $(D^2 - 2D + 1)y = x e^x \sin x$

Sol:

Given $f(D)y = Q$

A.E.M $m^2 - 2m + 1 = 0$
 $m = 1, 1$

$y_c = (C_1 + C_2 x) e^x$

Now $y_p = \frac{1}{D^2 - 2D + 1} x e^x \sin x$

$$= \frac{1}{(D-1)^2} e^x (x \sin x)$$

(a=1)

$$= e^x \frac{1}{(D+1-1)^2} x \sin x$$

$$= e^x \frac{1}{D^2} x \sin x.$$

$$= e^x \frac{1}{D} \left[\int x \sin x dx \right]$$

$$= e^x \frac{1}{D} \left[-x \cos x + \sin x \right]$$

$$= e^x \left[\int -x \cos x dx + \int \sin x dx \right]$$

$$= e^x \left[-x \sin x - \cos x - \cos x \right]$$

$$y_p = -e^x \left[x \sin x + 2 \cos x \right]$$

$$\therefore \text{G. sol}^n y = y_c + y_p$$

H.W.P :-

1) Solve $(D^2 + 2D + 1)y = x \cos x$

$$\frac{x}{2} \sin x + \frac{1}{2} (\cos x - \sin x)$$

2) Solve $\frac{d^2 y}{dx^2} + 9y = x \sin 2x$

$$\frac{x}{5} \sin 2x - \frac{6}{25} \cos 2x$$

3) Solve $(D^2 - 1)y = x^2 \cos x$

$$x \sin x + \frac{1}{2} (1 - x^2) \cos x$$

4) Solve $\frac{d^2 y}{dx^2} + y = x \sin x + (1 + x^2) e^x$

(*)

* METHOD OF VARIATION OF PARAMETERS:

This method using to find the General solution of only Second order linear D.E.ⁿ.

WORKING RULE:-

Let the given 2nd order L.D.E. $\frac{d^2y}{dx^2} + P\frac{dy}{dx} + Qy = R$ — ①

where P and Q are fn's of x (or) real constants and R is only a function of x .

Let it's C.F $y_c = c_1 u + c_2 v$, ^{of ①} where u and v are L.I. solutions which are fn's of x .

Now let P.I $y_p = Au + Bv$, ^{of ①} which is obtained from C.F by replacing c_1 by A and c_2 by B and A & B are fn's of x . which are given by

$$A = \int \frac{-vR}{uv'_1 - v'_1u} dx, \quad B = \int \frac{uR}{uv'_1 - v'_1u} dx$$

where $uv'_1 - v'_1u \neq 0$.

$\therefore$ Gr. solⁿ of eq ① is

$$y = y_c + y_p$$

$$\text{i.e., } y = c_1 u + c_2 v + Au + Bv.$$

PROBLEMS:-

1) Solve $y'' + y = \sec x$ by method of Variation of Parameters.

Sol: Given $(D^2 + 1)y = \sec x$. — (1)

$$f(D)y = R.$$

$$\text{where } f(D) = D^2 + 1, \quad R = \sec x.$$

$$\text{A.E.M } m^2 + 1 = 0 \Rightarrow m = \pm i$$

$$\text{C.F } y_c = C_1 \cos x + C_2 \sin x \quad \text{--- (2)}$$

$$= C_1 u + C_2 v$$

$$\text{where } u = \cos x, \quad v = \sin x.$$

$$u' = -\sin x, \quad v' = \cos x.$$

$$\therefore uv' - vu' = \cos^2 x + \sin^2 x = 1 \neq 0$$

$$\text{Now let P.I } y_p = A \cos x + B \sin x \quad \text{--- (3)}$$

$$\text{where } A = \int \frac{-vR}{uv' - vu'} dx, \quad B = \int \frac{uR}{uv' - vu'} dx$$

$$= \int \frac{-\sin x \cdot \sec x}{1} dx \quad = \int \frac{\cos x \cdot \sec x}{1} dx$$

$$= \int -\frac{\sin x}{\cos x} dx \quad = \int 1 dx$$

$$A = \log(\cos x) \quad B = x$$

$$y_p = \log(\cos x) \cdot \cos x + x \sin x$$

$$\therefore \text{Soln } y = y_c + y_p$$

$$y = C_1 \cos x + C_2 \sin x + \cos x \cdot \log(\cos x) + x \sin x$$

2) By using variation of parameters solve $y'' + 4y = \tan 2x$

Sol:

Given $(D^2 + 4)y = \tan 2x$ — (1)
 $f(D)y = R$

A.E.M $m^2 + 4 = 0 \Rightarrow m = \pm 2i$
 $\therefore y_c = C_1 \cos 2x + C_2 \sin 2x$ — (2)
 $= C_1 u + C_2 v$

where $u = \cos 2x, \quad v = \sin 2x$
 $u' = -2 \sin 2x \quad v' = 2 \cos 2x$

$uv' - vu' = 2 \cos^2 2x + 2 \sin^2 2x = 2 \neq 0$

Now let $y_p = A \cos 2x + B \sin 2x$ — (3)

where $A = \int \frac{-vR}{uv' - vu'} dx, \quad B = \int \frac{uR}{uv' - vu'} dx$
 $= \int \frac{-\sin 2x \cdot \tan 2x}{2} dx, \quad = \int \frac{\cos 2x \cdot \tan 2x}{2} dx$

$= \frac{1}{2} \int \frac{-\sin^2 2x}{\cos 2x} dx$

$= -\frac{1}{2} \int \frac{1 - \cos^2 2x}{\cos 2x} dx$

$= -\frac{1}{2} \left[\int \sec 2x dx - \int \cos 2x dx \right]$

$= -\frac{1}{2} \left[\frac{\log(\sec 2x + \tan 2x)}{2} - \frac{\sin 2x}{2} \right]$

$A = \frac{1}{4} \left[\sin 2x - \log(\sec 2x + \tan 2x) \right]$

From (3), $y_p = -\frac{1}{4} \cos 2x \cdot \log(\sec 2x + \tan 2x)$

$\therefore$ Soln $y = y_c + y_p$

③ Solve $(D^2+4)y = \sec 2x$ by the method of variation of parameters. (17)

Sol:

Given $f(D)y = R$ — (1), $R = \sec 2x$

A.E. $m^2+4=0 \Rightarrow m = \pm i2$

$y_c = C_1 \cos 2x + C_2 \sin 2x$ — (2)
 $= C_1 u + C_2 v$

$u = \cos 2x, \quad v = \sin 2x$

$uv' - vu' = 2 \neq 0$

let P.I $y_p = A \cos 2x + B \sin 2x$ — (3)

where $A = \int \frac{-\sin 2x \cdot \sec 2x}{2} dx, \quad B = \int \frac{\cos 2x \cdot \sec 2x}{2} dx$
 $= \frac{1}{2} \int -\frac{\sin 2x}{\cos 2x} dx \quad = \frac{1}{2} \int 1 dx$
 $= \frac{1}{2} \log(\cos 2x) \quad = \frac{1}{2} x$

$\therefore y_p = \frac{1}{4} \log(\cos 2x) \cdot \cos 2x + \frac{1}{2} x \sin 2x$

$\therefore \text{Sol'n } y = y_c + y_p$

4) Solve $(D^2-2D)y = e^x \sin x$ by M.V.P.

Sol:

Given $f(D)y = R, \quad m = 0, 2$

$y_c = C_1 + C_2 e^{2x} = C_1 u + C_2 v$

where $u=1, \quad v=e^{2x}$
 $u'=0, \quad v'=2e^{2x} \Rightarrow uv' - vu' = 2e^{2x}$

let $y_p = A + B e^{2x}$

$A = \int \frac{-e^{2x} \cdot e^x \sin x}{2e^{2x}} dx$

$= -\frac{1}{2} \int e^x \sin x dx$

$= -\frac{1}{2} \cdot \frac{e^x}{2} [\sin x - \cos x]$

$A = -\frac{e^x}{4} (\sin x - \cos x)$

$B = \int \frac{e^x \cdot e^x \sin x}{2e^{2x}} dx$

$= \frac{1}{2} \int e^{-x} \sin x dx$

$= \frac{1}{2} \cdot \frac{e^{-x}}{2} (-\sin x - \cos x)$

$= -\frac{e^{-x}}{4} (\sin x + \cos x)$

$\therefore y_p = -\frac{1}{2} e^x \sin x$

PRO H.W.P:-

1) Solve $\frac{d^2 y}{dx^2} + y = \sec x$ by M.V.P

2) Solve $(D^2 + a^2)y = \sec ax$

3) Solve $\frac{d^2 y}{dx^2} + y = 2 \sin x$

Q (4)

Solve $(D^2 + 1)y = \tan x$

Q (5)

Solve $(D^2 - 2D + 2)y = e^x \tan x$

(6)

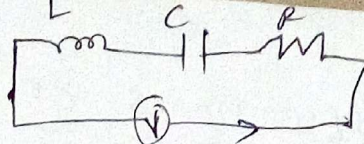
$y'' + 3y' + 2y = 2e^x$

$-x \cos x + \sin x \log x$

$\frac{\cos ax}{a^2} (\log(\cos ax) + \frac{x}{a} \sin ax)$

$\frac{x}{2} \sin x - \frac{x^2}{4} \cos x$

* (ii) L-C-R circuit:-



consider the discharge of a condenser C through an inductance L and the resistance R . Since the voltage drop across L , C and R are respectively $L \frac{d^2q}{dt^2}$, $\frac{q}{C}$ and $R \frac{dq}{dt}$.

By Kirchhoff's law, $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$

1) $q = c_1 e^{m_1 t} + c_2 e^{m_2 t}$

2) $q = (c_1 + c_2 t) e^{-\lambda t}$

3) $q = e^{-\lambda t} (c_1 \cos \mu t + c_2 \sin \mu t)$

(or) $\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = 0$

let $\frac{R}{L} = 2\lambda$, $\frac{1}{LC} = \mu^2$

$m = -\lambda \pm \sqrt{\lambda^2 - \mu^2}$

(i) $\lambda > \mu$ (ii) $\lambda = \mu$ (iii) $\lambda < \mu$

$(D^2 + 2\lambda D + \mu^2)q = 0$

* PROBLEM:-

1) A condenser of capacity C discharged through an inductance L and resistance R in series and the charge q at time t satisfies the eqn $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$.

Given that $L = 0.25$ henries, $R = 250$ ohms, $C = 2 \times 10^{-6}$ farads, and that when $t = 0$, charge q is 0.002 Coulombs and the current $\frac{dq}{dt} = 0$, obtain the value of q in terms of t .

Sol:-

Given D.Eqn $L \frac{d^2q}{dt^2} + R \frac{dq}{dt} + \frac{q}{C} = 0$

(or) $\frac{d^2q}{dt^2} + \frac{R}{L} \frac{dq}{dt} + \frac{1}{LC} q = 0$

Substitute given values

$\frac{d^2q}{dt^2} + \frac{250}{0.25} \frac{dq}{dt} + \frac{1}{(0.25)(2 \times 10^{-6})} q = 0$

$\frac{d^2q}{dt^2} + 1000 \frac{dq}{dt} + (2 \times 10^6) q = 0$

(or) $(D^2 + 1000D + 2 \times 10^6) q = 0$, $D \equiv \frac{d}{dt}$

A.E. $f(m) = 0 \Rightarrow m^2 + 1000m + 2 \times 10^6 = 0$

* APPLICATIONS OF DIFFⁿ EQ'S:-

* ELECTRICAL CIRCUITS:-

The formation of diffⁿ eqn for an electric circuit depends upon the following laws

Let 'i' be the current and 'q' the charge in the condenser plate at any time t. then

(i) $i = \frac{dq}{dt}$

(ii) Voltage drop across resistance (R) = Ri

(iii) Voltage drop across inductance (L) = $L \cdot \frac{di}{dt} = L \frac{dq}{dt}$

(iv) Voltage drop across capacitance (C) = $\frac{q}{C}$

* KIRCHHOFF'S LAW:-

1) Voltage law:- The algebraic sum of the voltage drops in each part of any closed electrical circuit is equal to the resultant "electromotive force" (e.m.f) in that circuit.

2) Current law:- At a junction or node, current coming is equal to current going.

(i) L-R circuit:-

Let i be the current flowing in the circuit containing resistance R and inductance L in series, with voltage source E, at any time t.

By voltage law, $\frac{di}{dt} + \frac{R}{L}i = \frac{E}{L}$ (L.D.E) $\frac{R}{L}$

It's solution $i = \frac{E}{R} + Ce^{-\frac{Rt}{L}}$ (i.e., i=0 at t=0)