UNIT-IV

MULTIPLE INTEGRALS

* DOUBLE INTEGRAL:-

Let $f(x,y)$ be a function defined at all points of a region R in the xy-plane, bounded by one or more curves. Let the region R be divided into small subregions each of area $\delta R_1, \delta R_2 \dots \delta R_n$ which are pairwise non overlapping. Let (x_i, y_i) be an arbitary point within the subregion δR_i.

Consider the sum.

$$f(x_1, y_1)\delta R_1 + f(x_2, y_2)\delta R_2 + \dots + f(x_n, y_n)\delta R_n.$$

If this sum tends to a finite limit as $n \to \infty$ such that the $\max(\delta R_i) \to 0$ irrespective of the choice of (x_i, y_i), the limit is called the "double integral" of $f(x,y)$ over the region R and is denoted by the symbol

$$\iint_R f(x,y)\,dx\,dy \quad (\text{or}) \quad \iint_R f(x,y)\,dR.$$

* PROPERTIES OF DOUBLE INTEGRAL

Let f and g are functions in x and y, defined and continuous in a region R.

(i) $\iint_R (f+g)\,dx\,dy = \iint_R f\,dx\,dy + \iint_R g\,dx\,dy$

(ii) $\iint_R Kf\,dx\,dy = K\iint_R f\,dx\,dy$

(iii) $\iint_R f\,dx\,dy = \iint_{R_1} f\,dx\,dy + \iint_{R_2} f\,dx\,dy$, $(R = R_1 \cup R_2)$

* A double or Triple integral is known as Multiple integral is an extension of Definite integral of a function of single variable to a function of two or more ~~three~~ variables.

* These are useful in evaluating area, volume, mass, centroid

in plane and solid regions.

* ITERATED INTEGRALS :-

An integral expression of the form $\int_a^b \int_{y_1(x)}^{y_2(x)} f(x,y)\,dy\,dx$ (or) $\int_a^b \int_{x_1(y)}^{x_2(y)} f(x,y)\,dx\,dy$ is called an iterated integral.

Infact both these are double integrals.

* DOUBLE INTEGRAL OVER RECTANGULAR :-

let $f(x,y)$ be ~~in~~ defined on reactangular region $R = \{(x,y) \mid a \le x \le b,\ c \le y \le d\}$ (or) $R = [a,b;c,d]$ then double integration of $f(x,y)$ over R defined as

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=a}^{x=b} f(x,y)\,dx \right] dy$$

(or)

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left[\int_{y=c}^{y=d} f(x,y)\,dy \right] dx.$$

Both Integrals are equal in Rectangal region.

DOUBLE INTEGRAL OVER NON-RECTANGULAR (GENERA REGIONS) :-

let $f(x,y)$ be a function defined over non-rectangular region R can be desecribe by two types of regions as.

(i) $R = \{(x,y) \mid a \le x \le b;\ y_1(x) \le y \le y_2(x)\}$ then

$$\iint_R f(x,y)\,dx\,dy = \int_{x=a}^{x=b} \left[\int_{y=y_1(x)}^{y_2(x)} f(x,y)\,dy \right] dx$$

and

(ii) $R = \{(x,y) \mid x_1(y) \le x \le x_2(y);\ c \le y \le d\}$ then

$$\iint_R f(x,y)\,dx\,dy = \int_{y=c}^{y=d} \left[\int_{x=x_1(y)}^{x=x_2(y)} f(x,y)\,dx \right] dy.$$

PROBLEMS :-

1) Evaluate $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx$

Sol:- $\int_0^3 \int_1^2 xy(1+x+y)\,dy\,dx = \int_0^3 \left[\int_1^2 (xy + x^2y + xy^2)\,dy \right] dx$

$$= \int_0^3 \left[x\frac{y^2}{2} + x^2\frac{y^2}{2} + x\frac{y^3}{3} \right]_{y=1}^{y=2} dx$$

$$= \int_0^3 \left[2x + 2x^2 + \frac{8x}{3} - \frac{x}{2} - \frac{x^2}{2} - \frac{x}{3} \right] dx$$

$$= \int_0^3 \left(\frac{3x^2}{2} + \frac{23x}{6} \right) dx$$

$$= \left(\frac{x^3}{2} + \frac{23}{12}x^2 \right)_0^3$$

$$= \frac{27}{2} + \frac{69}{4}$$

$$= \frac{123}{4}$$

2) Evaluate $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy$.

Sol: $\int_0^1 \int_x^{\sqrt{x}} (x^2+y^2)\,dx\,dy = \int_0^1 \left[\int_{y=x}^{y=\sqrt{x}} (x^2+y^2)\,dy \right] dx$

$$= \int_0^1 \left[x^2y + \frac{y^3}{3} \right]_{y=x}^{y=\sqrt{x}} dx$$

$$= \int_0^1 \left[x^{5/2} + \frac{x^{3/2}}{3} - x^3 - \frac{x^3}{3} \right] dx$$

$$= \left[\frac{x^{7/2}}{7/2} + \frac{x^{5/2}}{15/2} - \frac{4}{3}\frac{x^4}{4} \right]_0^1$$

$$= \frac{2}{7} + \frac{2}{15} - \frac{1}{3}$$

$$= \frac{9}{105}$$

$$= \frac{3}{35}$$

*3) Evaluate $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy$

Sol: $\int_0^5 \int_0^{x^2} x(x^2+y^2)\,dx\,dy = \int_0^5 \left[\int_0^{x^2} (x^3+xy^2)\,dy\right] dx$

$$= \int_0^5 \left(x^3y + \frac{xy^3}{3}\right)_0^{x^2} dx$$

$$= \int_0^5 \left(x^5 + \frac{x^7}{3}\right) dx$$

$$= \left(\frac{x^6}{6} + \frac{x^8}{24}\right)_0^5$$

$$= \frac{5^6}{6} + \frac{5^8}{24}$$

$$= 5^6\left[\frac{4+25}{24}\right]$$

$$= \frac{29}{24}(5^6)$$

4) Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx.$

Sol: $\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\,dy\,dx = \int_0^a \left[\int_0^{\sqrt{a^2-x^2}} \sqrt{(\sqrt{a^2-x^2})^2-y^2}\,dy\right] dx$

$$= \int_0^a \left[\frac{y}{2}\sqrt{a^2-x^2-y^2} + \frac{a^2-x^2}{2}\sin^{-1}\frac{y}{\sqrt{a^2-x^2}}\right]_0^{\sqrt{a^2-x^2}} dx$$

$$\left[\because \int\sqrt{a^2-x^2}\,dx = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a} + c\right]$$

$$= \int_0^a \left[0 + \frac{a^2-x^2}{2}\cdot\sin^{-1}(1)\right] dx$$

$$= \frac{1}{2}\cdot\frac{\pi}{2}\left[a^2x - \frac{x^3}{3}\right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3}\right]$$

$$= \frac{\pi a^3}{6}$$

5) $\int_0^1 \int_x^{\sqrt{x}} x^2y^2(x+3)\,dy\,dx$ $\left(\frac{18}{660}\right)$

* H.W.P :-

1) Evaluate $\int_{x=0}^{a}\int_{y=0}^{b}(x^2+y^2)\,dy\,dx$

2) Evaluate $\int_{0}^{a}\int_{0}^{\sqrt{a^2-y^2}}\sqrt{a^2-x^2-y^2}\,dx\,dy$

3) Evaluate $\int_{0}^{1}\int_{0}^{\sqrt{1+x^2}}\frac{dy\,dx}{1+x^2+y^2}$

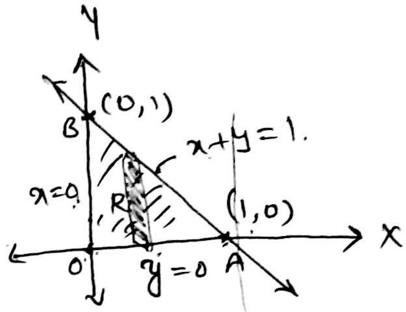
4) Evaluate $\int_{0}^{2}\int_{0}^{x} e^{x+y}\,dy\,dx$

5) what is the value of $\int_{0}^{2}\int_{0}^{x} y\,dy\,dx$. $4/3$

6) $\int_{0}^{4}\int_{y^2/4}^{y}\frac{y}{x^2+y^2}\,dx\,dy$ $(2\log 2)$ (7) $\int_{0}^{4}\int_{0}^{x^2} e^{y/x}\,dy\,dx$ $(3e^4-7)$

6) Evaluate $\iint (x^2+y^2)\,dx\,dy$ in the +ve quadrant for which $x+y\le 1$.

Sol: Given $\iint (x^2+y^2)\,dx\,dy$



From the graph,
we write limits
x limits $x=0,\ x=1$
y limits $y=0,\ y=1-x$
(OR) $y=0, y=1$ and $x=0, x=1-y$.

$$\therefore \iint_R (x^2+y^2)\,dx\,dy = \int_{x=0}^{x=1}\left[\int_{y=0}^{y=1-x}(x^2+y^2)\,dy\right]dx$$

$$= \int_0^1 \left[x^2y+\frac{y^3}{3}\right]_0^{1-x} dx$$

$$= \int_0^1 \left[x^2-x^3+\frac{(1-x)^3}{3}\right]dx$$

$$= \left[\frac{x^3}{3}-\frac{x^4}{4}+\frac{(1-x)^4}{-12}\right]_0^1$$

$$= \frac{1}{3}-\frac{1}{4}+\frac{1}{12}$$

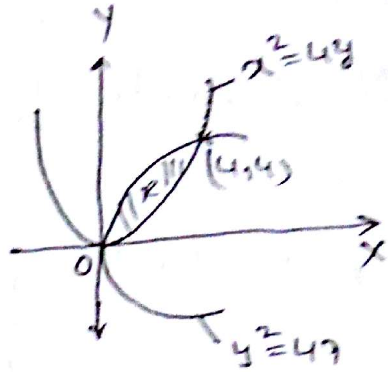
$$= \frac{1}{6}.$$

(7) Evaluate $\iint_R y\,dx\,dy$, where R is the region bounded by the parabolas $y^2 = 4x$ and $x^2 = 4y$.

Sol: Given $\iint_R y\,dx\,dy$

and Region by $y^2 = 4x$ and $x^2 = 4y$

$$x = \frac{y^2}{4} \Rightarrow \frac{y^4}{16} = 4y$$



$$y^4 - 64y = 0 \qquad \cancel{y^4 - 64y = 0}$$

$$y = 0,\ y = 4. \qquad \cancel{y = 0,}$$

So that $y = 0 \Rightarrow x = 0$

$y = 4 \Rightarrow x = 4$

$\therefore$ Both curves touches at $(0,0), (4,4)$

let The x limits $x = 0$ and $x = 4$

and y limits $y = \frac{x^2}{4}$ and $y = 2\sqrt{x}$

$\left[\because \frac{x^2}{4} \leq 2\sqrt{x},\ \forall x \in [0,4]\right]$

$$\therefore \iint_R y\,dx\,dy = \int_{x=0}^{x=4} \left[\int_{y=\frac{x^2}{4}}^{y=2\sqrt{x}} y\,dy\right] dx$$

$$= \int_0^4 \left(\frac{y^2}{2}\right)_{x^2/4}^{2\sqrt{x}} dx$$

$$= \int_0^4 \left(2x - \frac{x^4}{32}\right) dx$$

$$= \left(x^2 - \frac{x^5}{160}\right)_0^4$$

$$= 16 - \frac{4^5}{160}$$

$$= 16 - \frac{64 \times 16}{160}$$

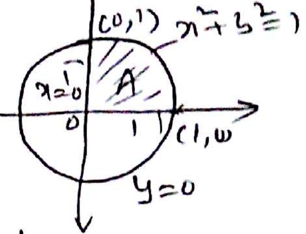
$$= 16 - \frac{32}{5}$$

$$= \frac{48}{5}$$

(8) Evaluate $\iint_A \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$, where A is the area in the +ve quadrant of the circle $x^2+y^2=1$.

Sol: Given $\iint_A \frac{xy}{\sqrt{1-y^2}}\,dx\,dy$, A is area in +ve quadrant of circle $x^2+y^2=1$.

From the graph, $y=0$, $y=1$ the limits are, and $x=0$, $x=\sqrt{1-y^2}$



$$\therefore \iint_A \frac{xy}{\sqrt{1-y^2}}\,dx\,dy = \int_{y=0}^{y=1}\left[\int_{x=0}^{x=\sqrt{1-y^2}} x\cdot\frac{y}{\sqrt{1-y^2}}\,dx\right]dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{1-y^2}} \cdot \frac{y}{\sqrt{1-y^2}}\,dy$$

$$= \int_0^1 \frac{1-y^2}{2}\cdot\frac{y}{\sqrt{1-y^2}}\,dy$$

$$= -\frac{1}{4}\int_0^1 \sqrt{1-y^2}\,.\,-2y\,dy$$

$$= -\frac{1}{4}\left(\frac{2\left(\sqrt{1-y^2}\right)^3}{3}\right)_0^1$$

$$= -\frac{1}{6}\,[0-1]$$

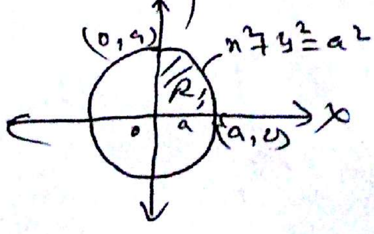
$$= \frac{1}{6}$$

* (9) Evaluate the double integral $\iint_R dx\,dy$, where R is in the first quadrant region ~~by~~ bounded by $x^2+y^2=a^2$

Sol: Given $\iint_R dx\,dy$, $R: x^2+y^2=a^2$ in I quadrant

limits are $x=0$, $x=a$.

$y=0$, $y=\sqrt{a^2-x^2}$



(1)

$$\therefore \iint_R dx\,dy = \int_{x=0}^{x=a} \int_{y=0}^{y=\sqrt{a^2-x^2}} dy\,dx$$

$$= \int_0^a [y]_0^{\sqrt{a^2-x^2}} dx$$

$$= \int_0^a \sqrt{a^2-x^2}\,dx$$

$$= \left[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\sin^{-1}\frac{x}{a}\right]_0^a$$

$$= \frac{a}{2}(0) + \frac{a^2}{2}\cdot\sin^{-1}(1) - 0 - 0$$

$$= \frac{a^2\pi}{4}$$

* H.W.P:

$x=0,\ y=0$ $\frac{2-x}{2}$

1) Evaluate $\iint_R xy\,dx\,dy$, where R is the region bounded by the line $x+2y=2$, lying in the first quadrant. $1/6$

2) Evaluate $\iint x^2\,dx\,dy$ over the region bounded by hyperbola $xy=4$, $y=0$, $x=1$, $x=4$. $x=1 \to 4$ $y=0, 4/x$

3) Evaluate $\int_0^1 \int_0^{\sqrt{1+x^2}} \dfrac{dy\,dx}{1+x^2+y^2}$. $\frac{\pi}{4}\log(1+\sqrt{2})$ (or) $\frac{\pi}{4}\sinh^{-1}1$

4) Evaluate $\iint xy(x+y)\,dx\,dy$ over the region R bounded by $y=x^2$ and $y=x$. $x=0 \to 1$ $y=x^2 \to x$ $3/56$

(5) Find the value of $\iint xy\,dx\,dy$ over the +ve quadrant of the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$. $\frac{a^2b^2}{8}$

$x=0 \to a$ $y=0 \to \frac{b}{a}\sqrt{a^2-x^2}$

$(a^4/3)$

$x=0 \to 2a$ $y=0 \to \frac{x^2}{4a}$

(6) $\iint_R xy\,dx\,dy$, R is by x-axis, ordinat $x=2a$, parve $x^2=4ay$

(7) $\iint xy\,dx\,dy$ over the +ve quadrant of the circle $x^2+y^2=a^2$ $(a^4/8)$

(7) $\iint (x^2+y^2)\,dx\,dy$ over the ~~region~~ area bounded by the ellipse (5)
$x^2/a^2 + y^2/b^2 = 1$ $\left[\begin{array}{l} x \to -a \to a \\ y = -\frac{b}{a}\sqrt{a^2-x^2} \to \frac{b}{a}\sqrt{a^2-x^2} \end{array} \right.$

* DOUBLE INTEGRAL IN POLAR COORDINATES:-

let $f(r,\theta)$ be a function in r and θ.

Then $\iint_R f(r,\theta)\,dr\,d\theta$, R is region.

let the limits of r in terms of θ. (functions)

i.e $r = f_1(\theta)$ to $f_2(\theta)$

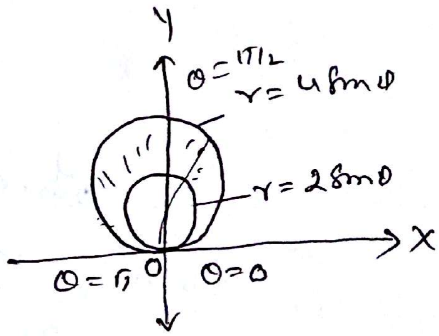
and θ limits $\theta_1 \to \theta_2$

$$\therefore \iint_R f(r,\theta)\,dr\,d\theta = \int_{\theta=\theta_1}^{\theta=\theta_2} \int_{r=f_1(\theta)}^{f_2(\theta)} f(r,\theta)\,dr\,d\theta.$$

PROBLEM:-

1) Evaluate $\iint r^3\,dr\,d\theta$ over the area included b/w the circles $r = 2\sin\theta$ and $r = 4\sin\theta$

Sol: Given $\iint r^3\,dr\,d\theta$



Region: $r = 2\sin\theta$, $r = 4\sin\theta$

from graph.,

r limits $r = 2\sin\theta$, $r = 4\sin\theta$

$\theta = 0$, $\theta = \pi$

$$\therefore \iint r^3\,dr\,d\theta = \int_{\theta=0}^{\theta=\pi} \int_{r=2\sin\theta}^{r=4\sin\theta} r^3\,dr\,d\theta = \int_0^{\pi} \left[\frac{r^4}{4}\right]_{2\sin\theta}^{4\sin\theta} d\theta$$

$$= \frac{1}{4}\int_0^{\pi} (256\sin^4\theta - 16\sin^4\theta)\,d\theta = \frac{1}{4}\int_0^{\pi} 240\sin^4\theta\,d\theta$$

$$= 60\int_0^{\pi} \sin^4\theta\,d\theta$$

$$= 60 \cdot 2\int_0^{\pi/2} \sin^4\theta\,d\theta$$

$$= 120 \times \frac{(4-1)}{4} \cdot \frac{(4-3)}{4-2} \cdot \frac{\pi}{2}$$

$$120 \times \frac{3}{4} \times \frac{1}{2} \times \frac{\pi}{2}$$

$$= 45\pi/2.$$

$\because \int_0^{2a} f(x)\,dx = 2\int_0^{a} f(x)\,dx$ [if $f(2a-x) = f(x)$]

$\int_0^{\pi/2} \sin^n x\,dx = \frac{n-1}{n} \cdot \frac{n-3}{n-2} \cdots \frac{1}{2} \cdot \frac{\pi}{2}$

n is even

* CHANGE OF VARIABLES IN DOUBLE INTEGRAL:-

Some times the evaluation of a double integral with it's present form may not be simple to evaluate. Then a given integral can be transformed into a simple integral involving the new variables.

* Transformation of Coordinates:-

let $F(x,y)$ be a given function in variables x and y.

Now let $x = f(u,v)$, $y = g(u,v)$ be the relations between the old variable (x,y) with the new variables (u,v) of the new coordinate system.

then $\iint_R F(x,y)\,dx\,dy = \iint_{R'} F(f(u,v), g(u,v))\,|J|\,du\,dv$

where $J = \dfrac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix}$

which is called the Jacobian of the coordinate transformation.

* Change of variables from Cartesian to Polar-cordinates:-

let $F(x,y)$ be function in x & y

let polar Coordinates $x = r\cos\theta$, $y = r\sin\theta$

$= f(r,\theta)$, $= g(r,\theta)$

and $J = \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos\theta & -r\sin\theta \\ \sin\theta & r\cos\theta \end{vmatrix}$.

$= r$

$\therefore\ dx\,dy \cong r\,dr\,d\theta$

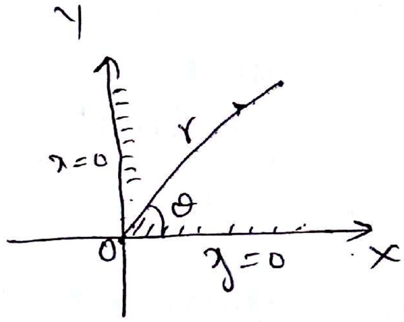
$\therefore \iint_R F(x,y)\,dx\,dy = \iint F(f(r,\theta), g(r,\theta))\,r\,dr\,d\theta$

(or) $= \int_{\theta=\theta_1}^{\theta_2} \int_{r=f_1(\theta)}^{r=f_2(\theta)} F(r\cos\theta, r\sin\theta)\,r\,dr\,d\theta$.

* PROBLEMS :-

1) Evaluate $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)}\, dx\, dy$ by changing to polar co-ordinates. Hence show that $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

Sol: Given $\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\, dy$



Here, In the integration

x limits are $x = 0$ to ∞

y " " $y = 0$ to ∞

changing to Polar coordinates

by putting $x = r\cos\theta,\ y = r\sin\theta$

$\Rightarrow x^2 + y^2 = r^2$

we have $dx\,dy = r\,dr\,d\theta$

Now, we have r limits are $r = 0$ to ∞

θ " " $\theta = 0$ to $\pi/2$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=\infty} e^{-r^2} . r\,dr\,d\theta$$

$$= \int_0^{\pi/2} \left(-\frac{1}{2} . e^{-r^2}\right)_0^\infty d\theta$$

$$= \int_0^{\pi/2} -\frac{1}{2}(-1)\, d\theta$$

$$= \frac{1}{2}(\theta)_0^{\pi/2}$$

$$\therefore \int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \frac{\pi}{4}.$$

Now $$\int_0^\infty \int_0^\infty e^{-(x^2+y^2)} dx\,dy = \left(\int_0^\infty e^{-x^2} dx\right)\left(\int_0^\infty e^{-y^2} dy\right)$$

$$= \left(\int_0^\infty e^{-x^2} dx\right)^2$$

Hence $\int_0^\infty e^{-x^2} dx = \frac{\sqrt{\pi}}{2}$.

② Evaluate $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx\, dy$, by transforming into a polar form.

Sol: Given $\int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx\, dy$

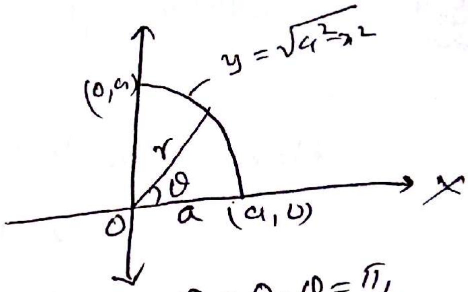
Here,

x limits are $x=0,\ x=a$

y " " $y=0,\ y=\sqrt{a^2-x^2}$

$\Rightarrow x^2+y^2=a^2$

given region is 1st quadrant of circle.



$\theta=0,\ \theta=\pi/2$

$r=0,\ r=a$

put $x=r\cos\theta,\ y=r\sin\theta$

$x^2+y^2=a^2$

$\therefore$ radius $r=a$

and we have $dx\,dy = r\,dr\,d\theta$

$$\therefore \int_0^a \int_0^{\sqrt{a^2-x^2}} y\sqrt{x^2+y^2}\, dx\, dy = \int_{\theta=0}^{\theta=\pi/2} \int_{r=0}^{r=a} r\sin\theta \cdot r \cdot r\,dr\,d\theta$$

$$= \int_{\theta=0}^{\theta=\pi/2} \left(\frac{r^4}{4}\right)_0^a \sin\theta\, d\theta$$

$$= \int_0^{\pi/2} \frac{a^4}{4} \cdot \sin\theta\, d\theta$$

$$= \frac{a^4}{4} \cdot (-\cos\theta)_0^{\pi/2}$$

$$= \frac{a^4}{4}(-0+1)$$

$$= \frac{a^4}{4}$$ //

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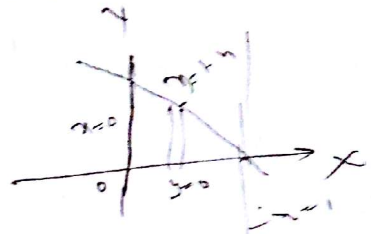
(3) Evaluate $\int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx$, by using the transformation $x+y=u$, $y=uv$.

Sol: Given $\int_0^1 \int_0^{1-x} e^{\frac{y}{x+y}}\, dy\, dx$

Here region $y=0, y=1-x$ and $x=0, x=1$



Given transformation

$x+y=u, \quad y=uv$

$\Rightarrow x+uv=u$

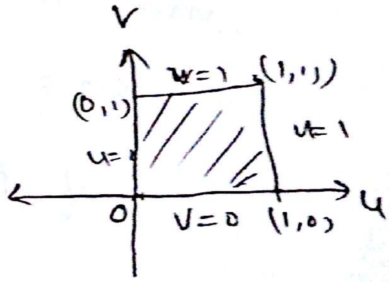
$\therefore x = u(1-v)$

Since $y=0 \Rightarrow uv=0 \Rightarrow u=0$ (or) $v=0$

and $y=1-x \Rightarrow x+y=1 \Rightarrow u=1$.

$x=0 \Rightarrow u(1-v)=0 \Rightarrow u=0$ (or) $v=1$

The given region transformed to a new region as $u=0, u=1$
$v=0, v=1$



$\therefore$ The Jacobian of transformation is given by

$$J = \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} \frac{\partial x}{\partial u} & \frac{\partial x}{\partial v} \\ \frac{\partial y}{\partial u} & \frac{\partial y}{\partial v} \end{vmatrix} = \begin{vmatrix} 1-v & -u \\ v & u \end{vmatrix}$$

$= u-uv+uv = u$

$\therefore dx\,dy = |J|\,du\,dv = u\,du\,dv$

$$\therefore \int_0^1 \int_0^{1-x} e^{y/(x+y)}\, dy\, dx = \int_{v=0}^{v=1} \int_{u=0}^{u=1} e^{uv/u}\, u\, du\, dv$$

$$= \int_0^1 e^v \cdot \left(\frac{u^2}{2}\right)_0^1 dv = \frac{1}{2}\left(e^v\right)_0^1$$

$$= \frac{1}{2}(e^1 - 1)$$

* H-W.P :-

① Evaluate double integral $\int_0^a \int_0^{\sqrt{a^2-y^2}} (x^2+y^2)\, dy\, dx$, using changing to Polar co-ordinates.

$r = 0 \to a$, $\theta = 0 - \frac{\pi}{2}$

$\frac{\pi a^4}{8}$

② Evaluate $\int_0^2 \int_0^{\sqrt{2x-x^2}} (x^2+y^2)\, dy\, dx$ by changing into Polar Co-ordinates

$r = 0 \to 2\cos\theta$

$\theta = 0 \to \frac{\pi}{2}$

$\frac{3\pi}{2}$

$x^2+y^2 = 2x$

$r^2 = 2r\cos\theta$

$y^2 = 2x - x^2$

$x^2+y^2 = 2x$

③ $\int_0^{4a} \int_{y^2/4a}^{y} \frac{x^2-y^2}{x^2+y^2}\, dx\, dy = 8a^2\left[\frac{\pi}{2} - \frac{5}{3}\right]$

$\theta \to 0 : \frac{\pi}{4} \to \frac{\pi}{2}$

$r = 0 \to \frac{4a\cos\theta}{\sin^2\theta}$

* PROBLEM

① By

* CHANGE OF ORDER OF INTEGRATION :-

change of order of integration implies that the change of limits of integration. If the region of integration consists of a vertical strip and slide along x-axis then in the changed order a horizontal strip and slide along y-axis are to be considered and vice-versa.

working rule :-

If the given integral $\int_{x=a}^{x=b} \int_{y=f_1(x)}^{f_2(x)} f(x,y)\, dy\, dx$.

Here In the region, x limits are $x=a$ & $x=b$ and y limits are $y=f_1(x)$ & $y=f_2(x)$,

Now order of changing i.e, y limits becomes into $y=a$, $y=b$ and x limits becomes into $x=f_1(y)$, $x=f_2(y)$. Then ~~the~~ after change of order of integral can be written as

$$\int_{y=a}^{y=b} \int_{x=f_1(y)}^{x=f_2(y)} f(x,y)\, dx\, dy.$$

* PROBLEMS:-

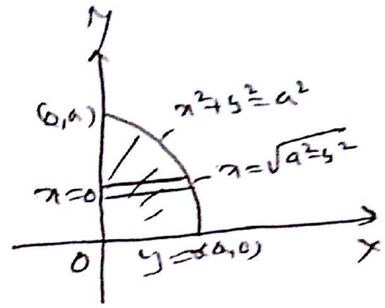
① By changing the order integration, evaluate

$$\int_0^a \int_0^{\sqrt{a^2-x^2}} \sqrt{a^2-x^2-y^2}\, dy\, dx.$$

Sol: Given region $x=0,\ x=a$ and $y=0,\ y=\sqrt{a^2-x^2}$ which is 1^{st} quadrant

$\Leftrightarrow y^2+x^2=a^2$

$\Leftrightarrow x=\sqrt{a^2-y^2}$



$\therefore$ After change the order, the limits $x=0,\ x=\sqrt{a^2-y^2}$

$y=0,\ y=a$

Hence, the integral becomes

$$\int_{y=0}^{y=a} \int_{x=0}^{x=\sqrt{a^2-y^2}} \sqrt{a^2-x^2-y^2}\, dx\, dy = \int_{y=0}^{a} \int_{x=0}^{\sqrt{a^2-y^2}} \sqrt{(\sqrt{a^2-y^2})^2-x^2}\, dx\, dy$$

$$= \int_0^a \left[\frac{x}{2}\sqrt{a^2-y^2-x^2} + \frac{a^2-y^2}{2} \sin^{-1}\frac{x}{\sqrt{a^2-y^2}} \right]_0^{\sqrt{a^2-y^2}} dy$$

$$= \int_0^a \left[0 + \frac{a^2-y^2}{2} \cdot \frac{\pi}{2} - 0 \right] dy$$

$$= \frac{\pi}{4}\left[a^2y - \frac{y^3}{3} \right]_0^a$$

$$= \frac{\pi}{4}\left[a^3 - \frac{a^3}{3} \right]$$

$$= \frac{\pi a^3}{6}$$

② Evaluate $\int_0^1 \int_0^{\sqrt{1-y^2}} x^3 y\, dx\, dy$, by changing the order of integration

Sol: Given region $y=0,\ y=1$

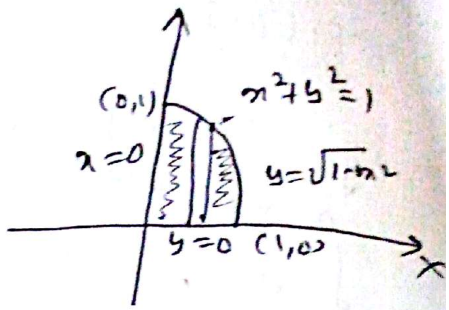
$x=0,\ x=\sqrt{1-y^2}$ in 1^{st} quadrant

$\Rightarrow y=\sqrt{1-x^2}$

After change the order,

y limits are $\quad y=0, \; y=\sqrt{1-x^2}$

x " " $\quad x=0, \; x=1$



Hence, the integral becomes.

$$\int_{x=0}^{1}\int_{y=0}^{\sqrt{1-x^2}} x^3 y^3 \, dy\, dx = \int_{x=0}^{1} x^3 \left[\frac{y^2}{2}\right]_0^{\sqrt{1-x^2}} dx$$

$$= \int_0^1 x^3 \frac{(1-x^2)}{2} dx$$

$$= \int_0^1 \frac{1}{2}(x^3 - x^5)\, dx$$

$$= \frac{1}{2}\left[\frac{x^4}{4} - \frac{x^6}{6}\right]_0^1$$

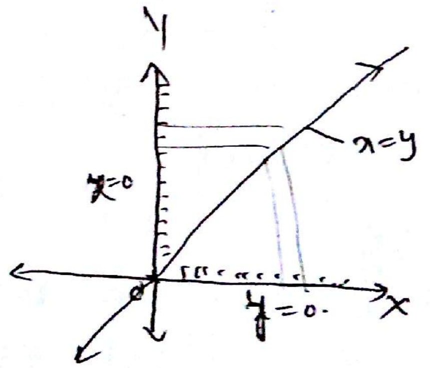
$$= \frac{1}{2}\left[\frac{1}{4} - \frac{1}{6}\right]$$

$$= \frac{1}{24}$$

(3) Evaluate the integral $\int_0^\infty \int_x^\infty \frac{e^{-y}}{y}\, dy\, dx$ by changing the order of integration.

Sol: Given region $\quad x=0, \; x=\infty$

$\quad y=x, \; y=\infty$



After, chang the order;

x limits $\quad x=0, \; x=y$

y limits $\quad y=0, \; y=\infty$

Hence the integral becomes

$$\int_{y=0}^{y=\infty}\int_{x=0}^{x=y} \frac{e^{-y}}{y}\, dx\, dy = \int_{y=0}^{y=\infty} \frac{e^{-y}}{y}(x)_0^y\, dy$$

$$= \int_0^\infty e^{-y}\, dy$$

$$= (-e^{-y})_0^\infty$$

$$= 1$$

(4) Evaluate by changing the order of the integration

$$\int_0^4\int_0^4 \frac{x}{x^2+y^2}\,dx\,dy.$$

Sol: Given region $x=0,\ x=4$ & $y=0,\ y=4$

Both limits are equal.

there is no changing limits.

Hence, the integral becomes.

$$\int_{x=0}^{x=4}\int_{y=0}^{y=4} \frac{x}{x^2+y^2}\,dy\,dx = \int_{x=0}^{4} x.\left(\frac{1}{x}.\,Tan^{-1}\frac{y}{x}\right)_0^4 dx$$

$$= \int_0^4 \left(Tan^{-1}\frac{4}{x} - 0\right) dx$$

$$= \int_{y=0}^{4} \frac{1}{2}.\left[\log(x^2+y^2)\right]_0^4 dy \qquad = \int_0^4 1.\,Tan^{-1}\frac{4}{x}\,dx$$

$$= \int_0^4 \frac{1}{2}.\left[\log(16+y^2) - \log y^2\right]dy = \left(x.\,Tan^{-1}\frac{4}{x}\right)_0^4 - \int_0^4 x.\frac{-4}{x^2}.\frac{x^2}{x^2+y^2}\,dx$$

$$\Rightarrow \frac{1}{2}\left[\log(16+y^2).y - \int_0^4 \frac{1}{16+y^2}.2y.y\right]$$

$$= 4.\frac{\pi}{4} - \frac{1}{4}\int_0^4 \frac{x^4}{x^2+4}\,dx$$

$$x^2+4\overline{)\,x^4\,}\ (x^2-4 + \frac{16}{x^2+4}$$

$$x^4+4x^2$$

$$-4x^2 + 16$$

$$-4x^2 - 16$$

$$16$$

$$= \pi - \frac{1}{4}\left[\int_0^4 \left((x^2-4) + \frac{16}{x^2+4}\right)dx\right]$$

$$= \pi - \frac{1}{4}\left[\frac{x^3}{3} - 4x + 16.\frac{1}{2}\,Tan^{-1}x/2\right]_0^4$$

$$= \pi - \frac{1}{4}\left[\frac{64}{3} - 16 + 8\,Tan^{-1}2\right]$$

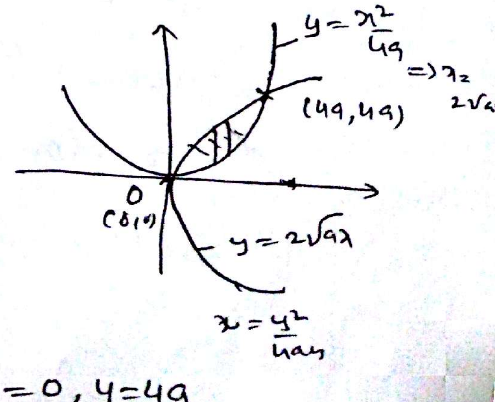
(5) change the order of integration and evaluate $\int_0^{4a}\int_{x^2/4a}^{2\sqrt{ax}} dy\,dx$

Sol: Given region $x=0,\ x=4a$

$y = \frac{x^2}{4a},\ y = 2\sqrt{ax}$

$\Rightarrow x=0 \Rightarrow y=0$ & $y=0$

$x=4a,\ y=4a$ & $y=4a$



After, change the order

$y = \frac{x^2}{4a} \Rightarrow x = 2\sqrt{ay}$

$y = 2\sqrt{ax} \Rightarrow x = \frac{y^2}{4a}$ and $y=0,\ y=4a$

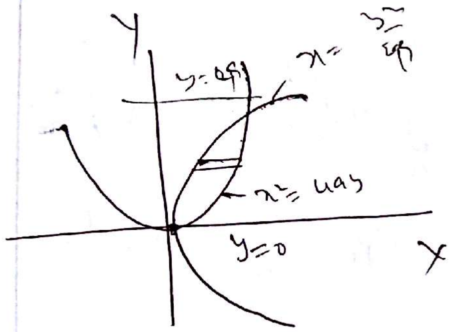
Here $\frac{y^2}{4a} \le x \le 2\sqrt{ay}$, $\forall y \in [0,4a]$

Hence, the integral becomes

$$\int_{y=0}^{y=4a}\int_{x=\frac{y^2}{4a}}^{x=2\sqrt{ay}} dx\,dy = \int_{y=0}^{y=4a}\left[x\right]_{\frac{y^2}{4a}}^{2\sqrt{ay}} dy$$

$$= \int_0^{4a}\left[2\sqrt{ay}-\frac{y^2}{4a}\right]dy$$

$$= \left[2\sqrt{a}\,\frac{y^{3/2}}{3/2}-\frac{y^3}{12a}\right]_0^{4a}$$



$$= \left[\frac{4\sqrt{a}}{3}\cdot(4a)^{3/2}-\frac{64a^3}{12a}\right]$$

$$= \cancel{\frac{32a^2}{3}}-\cancel{\frac{16a^2}{3}} \quad \frac{4}{3}\sqrt{a}\cdot 4a\sqrt{4a}-\frac{64a^3}{12a}$$

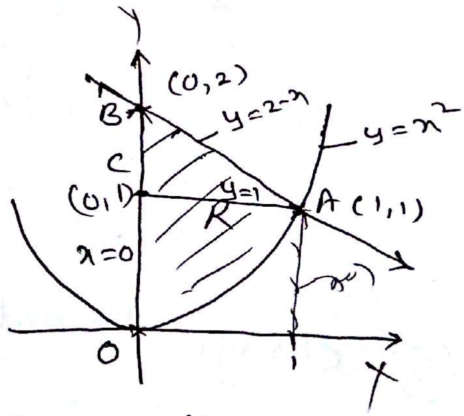
$$= \frac{32}{3}a^2-\frac{16}{3}a^2$$

$$= \frac{16}{3}a^2$$

⑥ By changing the order of integration, evaluate $\int_0^1\int_{x^2}^{2-x} xy\,dx\,dy$

Sol: Given region R:- $x=0$, $x=1$

and $y=x^2$, $y=2-x$.



$x=0 \Rightarrow y=2$ $(\because y=2-x)$

$x=1 \Rightarrow y=1$ $(\because y=2-x)$

and $x=0 \Rightarrow y=0$ $(\because y=x^2)$

$x=1 \Rightarrow y=1$

$x^2=2-x \Rightarrow x^2+x-2=0 \Rightarrow x=1,-2$ $\Rightarrow y=1,4$ $(1,1)$ & $(-2,4)$

Given Region OAB is bounded two sub regions, There are OAC & ACB.

After chang the order,

$y=x^2 \Rightarrow x=\sqrt{y}$

$y=2-x \Rightarrow x=2-y$

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(10)

The integral is after change the order

$$\iint_{OAB} = \iint_{OAC} + \iint_{ACB}$$

limits in OAC, $y=0, y=1$

$x=0, x=\sqrt{y}$

and limits in ACB, $y=1, y=2$

$x=0, x=2-y$

$\therefore$ The Integral becomes

$$\int_0^1 \int_{x^2}^{2-x} xy\, dx\, dy = \int_{y=0}^{y=1} \int_{x=0}^{x=\sqrt{y}} xy\, dx\, dy + \int_{y=1}^{y=2} \int_{x=0}^{x=2-y} xy\, dx\, dy$$

$$= \int_0^1 \left(\frac{x^2}{2}\right)_0^{\sqrt{y}} y\, dy + \int_1^2 \left(\frac{x^2}{2}\right)_0^{2-y} y\, dy$$

$$= \int_0^1 \frac{y}{2} \cdot y\, dy + \int_1^2 \frac{(2-y)^2}{2} \cdot y\, dy$$

$$= \left(\frac{y^3}{6}\right)_0^1 + \frac{1}{2}\left(2y^2 + \frac{y^4}{4} - \frac{4y^3}{3}\right)_1^2$$

$$= \frac{1}{6} + 4 + 2 - \frac{16}{3} - 1 - \frac{1}{8} + \frac{2}{3}$$

$$= \cancel{\frac{1+36-32}{6}} \quad \frac{1}{6} + 5 - \frac{1}{8} - \frac{14}{3}$$

$$= \frac{\cancel{4-3}\,24}{24}$$

$$= \frac{3}{8}$$

* H.W.P :-

1) Change the order of integration, evaluate $\int_0^a \int_{x/a}^{\sqrt{x/a}} (x^2+y^2)\,dx\,dy$

$y = 0 \to 1$

$x = ay^2 \to ay$

$\frac{a^3}{28} + \frac{a}{20}$

2) By changing the order of integration, evaluate

$$\int_0^3 \int_1^{\sqrt{4-y}} (x+y)\,dx\,dy$$

$241/60$

$x = 1 \to 2$

$y = 0 \to 4 - x^2$

3) By changing the order of integration, evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$

$y = 0 \to 1$

$x = 0 - \sqrt{1-y^2}$

$\pi/16$

4) By changing the order of integration, evaluate

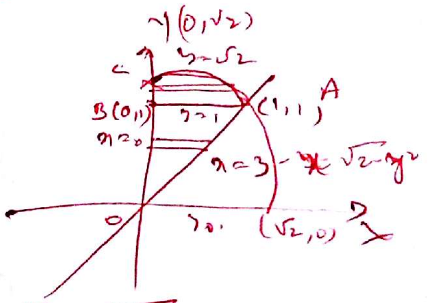
$$\int_0^1 \int_1^{2-x} xy\,dx\,dy$$

$5/24$

$y = 1 \to 2$

$x = 0 \to 2-y$

5) $$\int_0^1 \int_{x}^{\sqrt{2-x^2}} \frac{x\,dy\,dx}{\sqrt{x^2+y^2}}$$



$$= \int_{y=0}^{1} \int_0^{y} + \int_{y=1}^{\sqrt{2}} \int_0^{\sqrt{2-y^2}}$$

$$= 1 - \frac{1}{\sqrt{2}}$$

6) $$\int_0^1 \int_0^{\sqrt{1-x^2}} y^2\,dy\,dx$$

* TRIPLE INTEGRALS :

Let $f(x,y,z)$ be function defined over 3-dimensional finite region V. Now divide the region V into n elementary sub-divisions (volumes) $\delta v_1, \delta v_2 \ldots \delta v_n$. let (x_r, y_r, z_r) be any point in r-th sub division δv_r.

Consider the sum $\sum_{r=1}^{n} f(x_r, y_r, z_r)\delta v_r$

The limit of this sum, if it exists, as $n \to \infty$ and $\delta v_r \to 0$ is called the triple integral of $f(x,y,z)$ over the region V and is denoted by

$\iiint f(x,y,z)\, dv$ (or) $\iiint_V f(x,y,z)\, dx\, dy\, dz$.

* EVALUATION OF TRIPLE INTEGRAL:-

The triple integral is evaluated as the repeated integral $\int_{x_1}^{x_2}\int_{y_1}^{y_2}\int_{z_1}^{z_2} f(x,y,z)\, dx\, dy\, dz$

where the limits of z are z_1, z_2 which are either constants (or) functions of x and y; the limits of y are y_1, y_2 are either constants (or) functions of x. and the limits of x are x_1, x_2 which are constants.

The above integral also can be write

$$\int_{x_1}^{x_2}\left[\int_{y_1(x)}^{y_2(x)}\left[\int_{z_1(x,y)}^{z_2(x,y)} f(x,y,z)\, dz\right] dy\right] dx.$$

* PROBLEMS:-

1) Evaluate $\int_{z=0}^{1}\int_{y=1}^{2}\int_{x=2}^{3} xyz\,dx\,dy\,dz$

Sol: $\int_0^1\int_1^2\int_2^3 xyz\,dx\,dy\,z = \int_0^1\int_1^2 \left(\frac{x^2}{2}\right)_2^3 yz\,dy\,dz$

$= \int_0^1 \left(\frac{\cancel{9}}{2} - \frac{4}{2}\right)\left(\frac{y^2}{2}\right)_1^2 z\,dz$

$= \frac{\cancel{9}5}{2}\left(\frac{4}{2} - \frac{1}{2}\right)\left(\frac{z^2}{2}\right)_0^1$

$= \frac{5}{2}\cdot\frac{3}{2}\cdot\frac{1}{2}$

$= \frac{15}{8}$ //

2) Evaluate $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx$.

Sol: $\int_0^1\int_0^{\sqrt{1-x^2}}\int_0^{\sqrt{1-x^2-y^2}} xyz\,dz\,dy\,dx = \int_0^1\int_0^{\sqrt{1-x^2}} xy\left(\frac{z^2}{2}\right)_0^{\sqrt{1-x^2-y^2}} dy\,dx$

$= \int_0^1\int_0^{\sqrt{1-x^2}} xy\left[\frac{1-x^2-y^2}{2}\right] dy\,dx$

$= \frac{1}{2}\int_0^1\int_0^{\sqrt{1-x^2}} (xy - x^3y - xy^3)\,dy\,dx$

$= \frac{1}{2}\int_0^1 \left[x\frac{y^2}{2} - x^3\frac{y^2}{2} - x\frac{y^4}{4}\right]_0^{\sqrt{1-x^2}} dx$

$= \frac{1}{2}\int_0^1 \left[\frac{1}{2}(x-x^3) - \frac{(x^3-x^5)}{2} - \frac{1}{4}(x+x^5-2x^3)\right]dx$

$= \frac{1}{8}\int_0^1 [2x - 2x^3 - \cancel{2x^3} + 2x^5 - x - x^5 + \cancel{2x^3}]\,dx$

$= \frac{1}{8}\int_0^1 (x^5 - 2x^3 + x)\,dx = \frac{1}{8}\left[\frac{x^6}{6} - \frac{x^4}{2} + \frac{x^2}{2}\right]_0^1$

$= \frac{1}{48}$ // $= \frac{1}{8}\left[\frac{1}{6} - \frac{1}{2} + \frac{1}{2}\right] = \frac{1}{8}\left[\cancel{\frac{2-3+6}{12}}\right] = \cancel{\frac{5}{96}}$ //

③ Evaluate $\int\limits_{x=0}^{1}\int\limits_{y=0}^{1}\int\limits_{z=\sqrt{x^2+y^2}}^{2} xyz\; dz\,dy\,dx$

Sol:

$$\int\limits_{x=0}^{1}\int\limits_{y=0}^{1}\int\limits_{z=\sqrt{x^2+y^2}}^{2} xyz\; dz\,dy\,dx = \int\limits_{0}^{1}\int\limits_{0}^{1} xy\left[\frac{z^2}{2}\right]_{\sqrt{x^2+y^2}}^{2} dy\,dx$$

$$= \int\limits_{0}^{1}\int\limits_{0}^{1} xy\left[\frac{4}{2} - \frac{x^2+y^2}{2}\right] dy\,dx$$

$$= \frac{1}{2}\int\limits_{0}^{1}\int\limits_{0}^{1} x\left[4y - x^2y - y^3\right] dy\,dx$$

$$= \frac{1}{2}\int\limits_{0}^{1} x\left[\frac{4y^2}{2} - x^2\frac{y^2}{2} - \frac{y^4}{4}\right]_0^1 dx$$

$$= \frac{1}{2}\int\limits_{0}^{1} x\left[\frac{4}{2} - x^2\frac{1}{2} - \frac{1}{4}\right] dx$$

$$= \frac{1}{8}\int\limits_{0}^{1}\left[8x - 2x^3 - x\right] dx$$

$$= \frac{1}{8}\left[7\frac{x^2}{2} - \frac{x^4}{2}\right]_0^1$$

$$= \frac{1}{8}\left[\frac{7}{2} - \frac{1}{2}\right]$$

$$= \frac{3}{8}$$

4) Evaluate $\iiint xyz\; dz\,dy\,dx$ over the volume enclosed by three co-ordinate plane and the plane $x+y+z=1$.

(or)

Evaluate $\int\limits_{x=0}^{x=1}\int\limits_{y=0}^{y=1-x}\int\limits_{z=0}^{1-x-y} xyz\; dz\,dy\,dx.$

Sol: the volume enclosed by three co-ordinate plane i.e, $x=0,\ y=0,\ z=0$ and given plane $x+y+z=1$

let $z = 1-x-y$ and $z=0$

$y=0,\quad y = 1-x \quad (\because z=0)$

$x=0,\quad x=1.\ (\because y=0, z=0)$

$$= \frac{1}{2}\int_0 x\left[\frac{1}{12}(1-x)^4\right]dx$$

$$= \frac{1}{24}\int_0^1 x(1+x^2-2x)^2\,dx$$

$$= \frac{1}{24}\int_0^1 [x+x^5+4x^3+2x^3-4x^4-4x^2]\,dx$$

$$= \frac{1}{24}\left[\frac{x^2}{2}+\frac{x^6}{6}+6\frac{x^4}{4}-4\frac{x^5}{5}-4\frac{x^3}{3}\right]_0^1$$

$$= \frac{1}{24}\left[\frac{1}{2}+\frac{1}{6}+\frac{3}{2}-\frac{4}{5}-\frac{4}{3}\right]$$

$$= \frac{1}{24}\left[2+\frac{1}{6}-\frac{4}{5}-\frac{4}{3}\right]$$

$$= \frac{1}{24}\cdot\frac{60+5-24-40}{30}$$

$$= \frac{1}{24}\cdot\frac{1}{30}$$

$$= \frac{1}{720}$$ //

(7) $\iiint_V dx\,dy\,dz$, whe
V is $x=0, y=0, z=0$ & $2x+3y+4z=12$
(12)

* H.W.P:-

1) Evaluate $\int_0^1\int_y^1\int_0^{1-x} x\,dz\,dx\,dy$ $\frac{1}{12}$

2) Evaluate $\int_0^1\int_0^{1-x}\int_0^{1-x-y} e^x\,dx\,dy\,dz$ $\frac{1}{2}(e-3)$

3) Evaluate $\iiint_V (xy+yz+zx)\,dx\,dy\,dz$, where V is the region of space bounded by $x=0, x=1, y=0, y=2, z=0, z=3$ $(33/2)$

4) Evaluate $\int_0^1\int_0^1\int_0^1 (x^2+y^2+z^2)\,dz\,dy\,dx$ 1

5) Evaluate $\iiint xy^2z\,dx\,dy\,dz$ over the positive octant of the sphere $x^2+y^2+z^2=a^2$. $\left(\frac{a^7}{105}\right)$

(6) $\int_0^{\log 2}\int_0^x\int_0^{x+y} e^{x+y+z}\,dz\,dy\,dx$ $(5/8)$

* CHANGE OF VARIABLES IN A TRIPLE INTEGRAL :-

let the functions $x=\phi_1(u,v,w)$, $y=\phi_2(u,v,w)$ and $z=\phi_3(u,v,w)$ be the transformation from cartesian coordinates to the curvilinear coordinates u, v, w.

The Jacobian for this transformation is given by $J=\dfrac{\partial(x,y,z)}{\partial(u,v,w)}$

Then $\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(\phi_1,\phi_2,\phi_3)\,|J|\,du\,dv\,dw$.

where V' is the corresponding domain in the curvilinear coordinates u,v,w.

*(1) change of variables from cartesian to spherical Polar co-ordinate system:-

The relation b/w the cartesian coordinates x,y,z and Spherical Polar Coordinates r, θ, ϕ are given by

$$\boxed{\begin{aligned} x &= \rho\sin\theta\cos\phi \\ y &= \rho\sin\theta\sin\phi \\ z &= \rho\cos\theta \end{aligned}}$$

(or) $\rho\cos\theta\sin\phi$
(or) $\rho\sin\theta\sin\phi$
(or) $\rho\cos\phi$

$x = r\sin\phi\cos\theta$
$y = r\sin\phi\sin\theta$
$z = r\cos\phi$
$(0\le\theta\le 2\pi, 0\le\phi\le\pi)$

and $dx\,dy\,dz = |J|\,dr\,d\theta\,d\phi$

where Jacobian of (x,y,z)

$$J=\frac{\partial(x,y,z)}{\partial(\rho,\theta,\phi)} = \rho^2\sin\phi. \text{ (or) } r^2\sin\phi$$

Then, $\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\sin\theta\cos\phi, r\sin\phi\sin\phi, r\cos\theta)\,\rho^2\sin\phi\,d\rho\,d\theta\,d\phi$.

* (ii) change of variables from Cartesian to cylindrical co-ordinate system :-

The relations b/w the Cartesian coordinates x, y, z and the cylindrical coordinates r, θ, z are given by

$$x = r\cos\theta$$
$$y = r\sin\theta$$
$$z = z$$

and $dx\,dy\,dz = |J|\,dr\,d\theta\,dz$

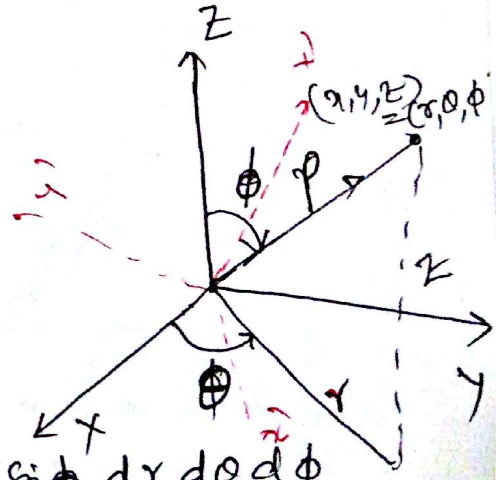
Jacobian of (x,y,z) is

$$J = \frac{\partial(x,y,z)}{\partial(r,\theta,z)} = r$$

Thus $\iiint_V f(x,y,z)\,dx\,dy\,dz = \iiint_{V'} f(r\cos\theta, r\sin\theta, z)\, r\,dr\,d\theta\,dz.$

PROBLEMS :-

1) Evaluate $\int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}}$ by changing to spherical polar coordinates.



Sol: Let Spherical Polar Coordinates

$x=0,\ x=1$
$y=0,\ y=\sqrt{1-x^2}$
$z=0,\ z=\sqrt{1-x^2-y^2}$

$$x = \rho\sin\phi\cos\theta$$
$$y = \rho\sin\theta\sin\phi$$
$$z = \rho\cos\phi$$

$0 \le r \le 1$.

$1 + z^2 =$
$z = 0$

$\cos\theta = \cos\frac{\pi}{2}$
$\theta = \frac{\pi}{2}$

and $dx\,dy\,dz = \rho^2 \sin\phi\,dr\,d\theta\,d\phi$

and $x^2+y^2+z^2 = \rho^2$

$r \ge 0$

Given region is lies in the first (+ve) ~~qua~~ octant of $x^2+y^2+z^2=1$.

$y=1=$

$\rho = 0 \to 1$

$\therefore$ limits of ρ varies from 0 to 1

θ varies from 0 to $\pi/2$

ϕ " " 0 to $\pi/2$

$$\therefore \int_0^1 \int_0^{\sqrt{1-x^2}} \int_0^{\sqrt{1-x^2-y^2}} \frac{dx\,dy\,dz}{\sqrt{1-x^2-y^2-z^2}} = \int_{\phi=0}^{\pi/2} \int_{\theta=0}^{\pi/2} \int_{\rho=0}^{1} \frac{1}{\sqrt{1-r^2}}\, r^2 \sin\phi \; dr\,d\theta\,d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\phi \left(\int_{r=0}^{r=1} \frac{1-(1-r^2)}{\sqrt{1-r^2}}\, dr \right) d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[\int_0^1 \frac{1}{\sqrt{1-r^2}} - \sqrt{1-r^2} \right] dr\, d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[\sin^{-1} r - \left[\frac{r}{2}\sqrt{1-r^2} + \frac{1}{2}\sin^{-1} r \right] \right]_0^1 d\theta\, d\phi$$

$$= \int_0^{\pi/2} \int_0^{\pi/2} \sin\theta \left[\frac{\pi}{2} - \frac{\pi}{4} \right] d\theta\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} (-\cos\theta)_0^{\pi/2}\, d\phi$$

$$= \frac{\pi}{4} \int_0^{\pi/2} (1)\, d\phi$$

$$= \frac{\pi}{4} (\phi)_0^{\pi/2}$$

$$= \frac{\pi^2}{8} \;//$$

① Evaluate $\iiint (x^2+y^2+z^2)\, dx\,dy\,dz$ taken over the volume enclosed by the sphere $x^2+y^2+z^2=1$, by transforming into spherical polar coordinates

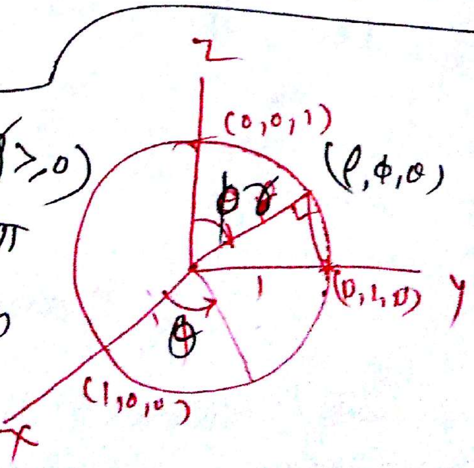
$(4\pi/5)$

$\cancel{r}\,\rho$ = distance to origin ($\rho \geq 0$)

ϕ = angle to z-axis, $0 \leq \phi \leq \pi$

θ $\cancel{\phi}$ = usual ϕ = angle projection to xy-plane with x-axis

$0 \leq \theta \leq 2\pi$



APPLICATIONS OF TRIPLE INTEGRALS

CALCULATION OF VOLUMES:

The volume of 3-dimensional solid is $\iiint_V dv = \iiint_V dx\,dy\,dz$ where the integration is carried over the entire volume.

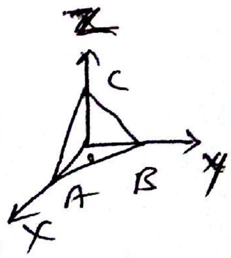
PROBLEMS:-

1) Find the volume of the tetrahedron bounded by the planes $x=0$, $y=0$, $z=0$ and $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$.

(OR)

Find the volume of the tetrahedron bounded by the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$ and the coordinate planes by triple integral.

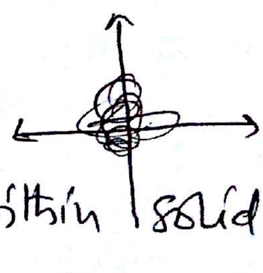
Sol: The required volume $= \iiint_V dx\,dy\,dz$



on the plane $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$

$$z = c\left[1-\frac{x}{a}-\frac{y}{b}\right]$$

Hence, for a fixed ~~point~~ (x,y) on the xy plane within ΔOAB,



z varies from 0 to $c\left[1-\frac{x}{a}-\frac{y}{b}\right]$ within solid

Then for a fixed x within the ΔOAB, y varies from 0 to $b\left(1-\frac{x}{a}\right)$.

Then x varies from 0 to a.

∴ The required volume of the tetrahedron

$$= \int_{x=0}^{x=a} \int_{y=0}^{y=b(1-\frac{x}{a})} \int_{z=0}^{c(1-\frac{x}{a}-\frac{y}{b})} dz\,dy\,dx$$

$$= \int_0^a \int_0^{b(1-\frac{x}{a})} c\left[1-\frac{x}{a}-\frac{y}{b}\right] dy\,dx$$

$$= \int_0^a c\left[y-\frac{xy}{a}-\frac{y^2}{2b}\right]_0^{b(1-\frac{x}{a})} dx$$

$$= \int_0^a c\left[(1-\frac{x}{a})y-\frac{1}{2b}y^2\right]_0^{b(1-\frac{x}{a})} dx$$

$$= \int_0^a \left[cb\left(1-\frac{x}{a}\right)^2-\frac{cb}{2}\left(1-\frac{x}{a}\right)^2\right] dx$$

$$= \cancel{\frac{cb\,(1-\frac{x}{a})^3}{(-3/a)}} - \cancel{\frac{cb}{2}\cdot(1-\frac{x}{a})}$$

$$= \int_0^a \frac{cb}{2}\left[1-\frac{x}{a}\right]^2 dx$$

$$= \left[\frac{cb}{2}\cdot\frac{(1-\frac{x}{a})^3}{(-3/a)}\right]_0^a$$

$$= \frac{abc}{6} \text{ cubic units.}$$

② Evaluate $\iiint (x+y+z)\,dx\,dy\,dz$ over the tetrahedron bounded by the co-ordinate planes and the plane $x+y+z=1$ ($\frac{1}{4}$)

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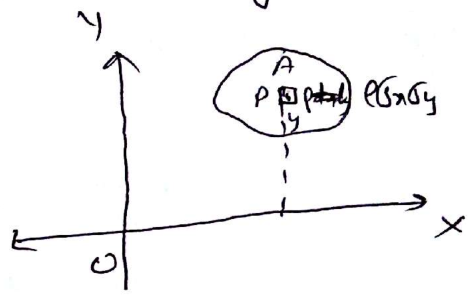
① $\int_0^{\pi/2} \int_0^{a\sin\theta} \int_0^{\frac{a^2-r^2}{2}} r\,dz\,dr\,d\theta$ $\left(\frac{5\pi a^4}{128}\right)$

8) *

MOMENT OF INERTIA:

If a particle of mass m of a body be at a distance r from a given line, then "mr^2" is called the moment of inertia of the particle about the given line and the sum of similar expressions taken for all the particles of the body, i.e, Σmr^2 is called the moment of inertia of the body about the given line.



If M be the total mass of the body and we write its moment of inertia $= Mk^2$, then K is called the radius of gyration of the body about the axis.

M.I of a plane lamina :-

Consider an elementary mass $\rho\,\delta x\,\delta y$ at the point $P(x,y)$ of a plane area A so that it's M.I about x-axis $= \rho\,\delta x\,\delta y\,y^2$

$\therefore$ M.I of the lamina about x-axis i.e

$$I_x = \iint_A \rho y^2\,dx\,dy$$

lly " " "

$$I_y = \iint_A \rho x^2\,dx\,dy$$

Also M.I of the lamina about an axis $\perp^r$ to the xy-plane i.e $I_z = \iint_A \rho(x^2+y^2)\,dx\,dy.$

M.I of a solid:-

consider an elementary mass $\rho\,\delta x\,\delta y\,\delta z$ enclosing a point $P(x,y,z)$ of a solid of volume V

Distance of P from the x-axis $= \sqrt{x^2+y^2}$

$\therefore$ M.I of this element about the x-axis

$$= \rho\,\delta x\,\delta y\,\delta z\,(x^2+y^2)$$

Thus M.I of this solid about x-axis i.e.,

$$I_x = \iiint_V \rho(y^2+z^2)\,dx\,dy\,dz$$

lly " " " $\quad I_y = \iiint_V \rho(z^2+x^2)\,dx\,dy\,dz$

" " " $\quad I_z = \iiint_V \rho(x^2+y^2)\,dx\,dy\,dz.$

(1) Find the moment of inertia about z-axis of the region in the first octant bounded by $\frac{x}{a}+\frac{y}{b}+\frac{z}{c}=1$,

$$M.I = \int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})}\int_{z=0}^{c[1-\frac{x}{a}-\frac{y}{b}]} \rho(x^2+y^2)\,dz\,dy\,dx$$

$$= \rho\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} (x^2+y^2)\,[1]_0^{c(1-\frac{x}{a}-\frac{y}{b})}\,dy\,dx$$

$$= \rho\int_{x=0}^{a}\int_{y=0}^{b(1-\frac{x}{a})} c(x^2+y^2)\left(1-\frac{x}{a}-\frac{y}{b}\right)dy\,dx$$