

EM-I

UNIT-I

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DIFFERENTIAL CALCULUS

Functions of Single Variables

*Continuous function on closed interval:-

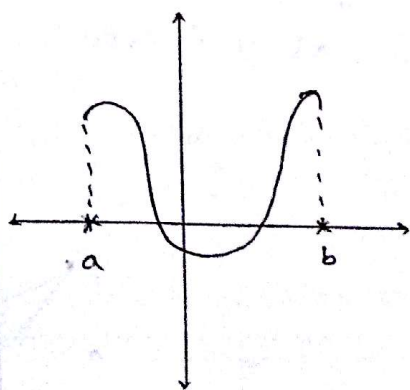
The function $f: [a, b] \rightarrow \mathbb{R}$ is said to be continuous on closed interval $[a, b]$ if

(i) $\lim_{x \rightarrow c} f(x) = f(c), \forall c \in (a, b)$
i.e, f is continuous at $c \in (a, b)$

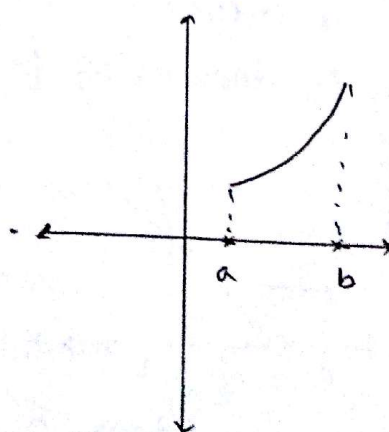
(ii) $\lim_{x \rightarrow a^+} f(x) = f(a)$
i.e, f is Right continuous at 'a'

(iii) $\lim_{x \rightarrow b^-} f(x) = f(b)$
i.e, f is left continuous at 'b'

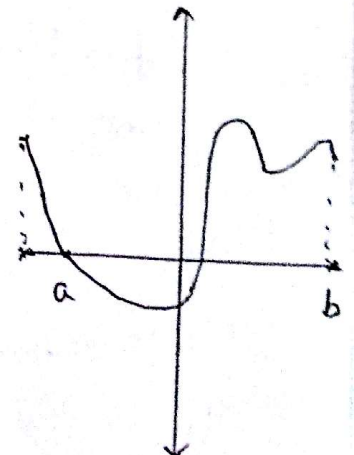
Geometrically, if $f(x)$ is a continuous function in the closed interval $[a, b]$, the graph of $y = f(x)$ is a continuous curve for the points x in $[a, b]$.



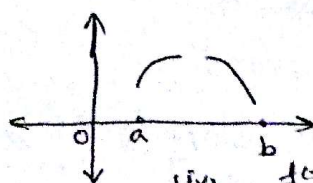
(i)



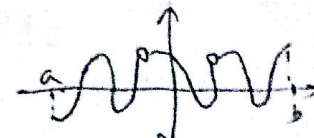
(ii)



(iii)



(iv)



$f(x)$ is not continuous (discontinuous)

* Differentiable function on closed interval:-

The function $f: [a, b] \rightarrow \mathbb{R}$ is said to be differentiable function on closed interval $[a, b]$ if

(i) $f'(c) = \lim_{x \rightarrow c} \frac{f(x) - f(c)}{x - c}$ exists, where $\forall c \in (a, b)$.

i.e, f is derivable function at $\forall c \in (a, b)$.

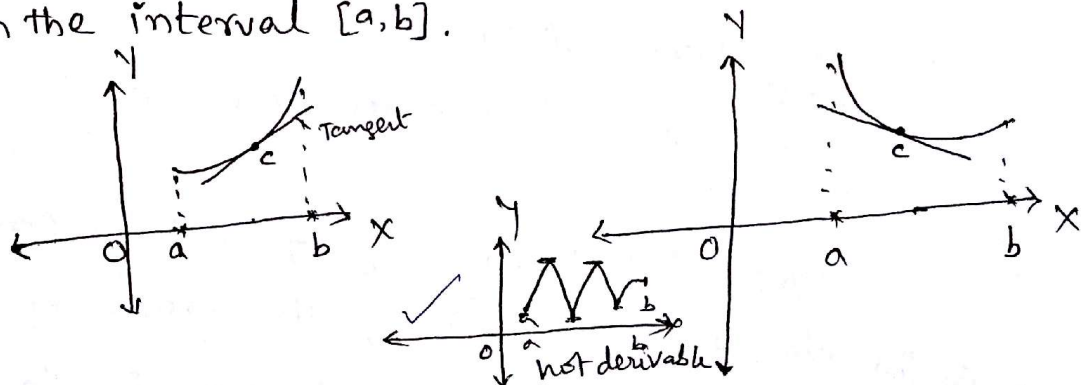
(ii) $Rf'(a) = \lim_{x \rightarrow a^+} \frac{f(x) - f(a)}{x - a}$ exists.

i.e, f is right derivable function at 'a'

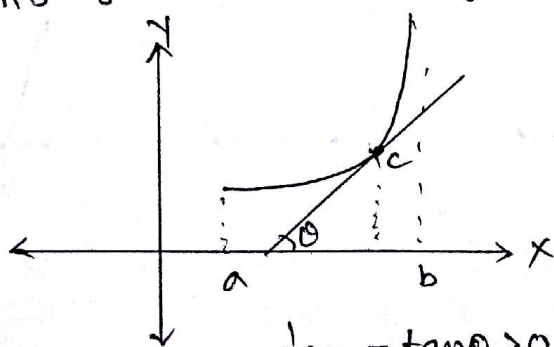
(iii) $Lf'(b) = \lim_{x \rightarrow b^-} \frac{f(x) - f(b)}{x - b}$ exists

i.e, f is left derivable function at 'b'.

Geometrically, if $f(x)$ is derivable on $[a, b]$ there exists a unique tangent to the curve at every point in the interval $[a, b]$.

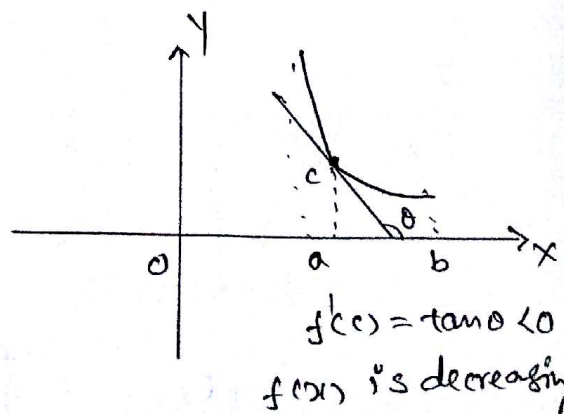


/* If $f'(c)$ is +ve, it means that $f(x)$ is increasing function as x increases via c . If $f'(c)$ is -ve, it means $f(x)$ is decreasing function as x increases via c .



$$f'(c) = \tan \theta > 0$$

$f(x)$ is increasing



$$f'(c) = \tan \theta < 0$$

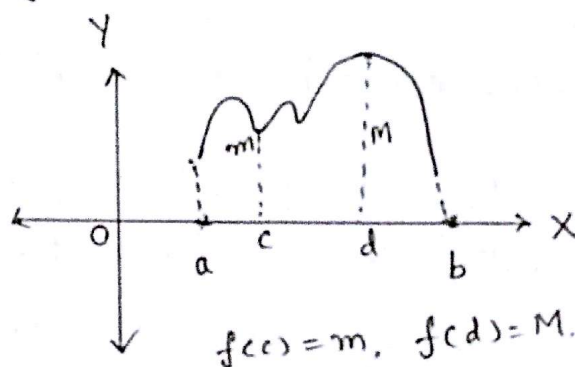
$f(x)$ is decreasing

Here, we recall that $f'(c)$ is slope of tangent to the curve $y = f(x)$ at $x = c$.

* Properties of Continuous functions:

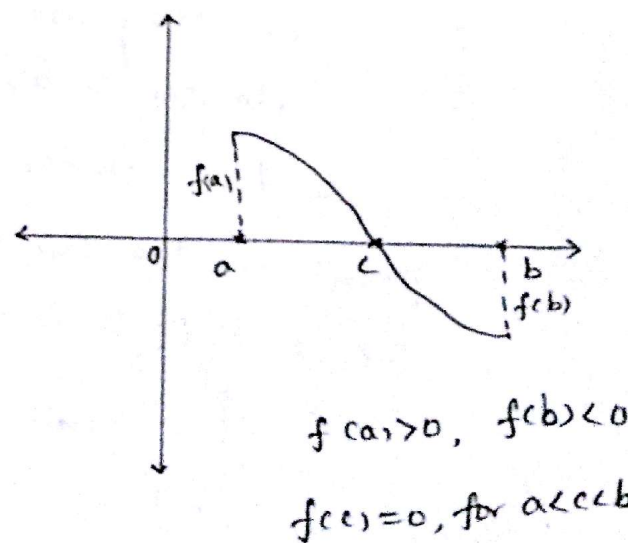
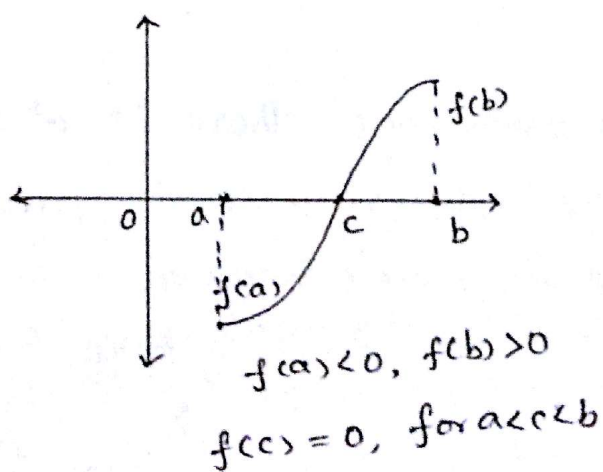
(1) If $f(x)$ is continuous in $[a, b]$, then $f(x)$ is bounded in $[a, b]$ and also it attains its greatest lower bound (infimum) and the least upper bound (Supremum).

that means, if m is infimum and M is supremum of $f(x)$ in $[a, b]$ there exists points c and d in $[a, b]$ such that $f(c) = m$, $f(d) = M$



(2) If $f(x)$ is continuous in $[a, b]$, it attains all values between $f(a)$ and $f(b)$.

(3) If $f(x)$ is continuous in $[a, b]$ and, $f(a)$ and $f(b)$ are opposite signs, then there exists at least one point c in (a, b) such that $f(c) = 0$.



MEAN VALUE THEOREMS:

(I) ROLLE'S THEOREM:

Statement:- Let $f: [a, b] \rightarrow \mathbb{R}$ be a function such that
(i) $f(x)$ is continuous on $[a, b]$ (ii) $f(x)$ is differentiable in (a, b) and (iii) $f(a) = f(b)$. Then there exists at least one point c in (a, b) such that $f'(c) = 0$.

Geometric Interpretation of Rolle's theorem:-

Geometrically, The function $f(x)$ satisfying three conditions of Rolle's theorem. Then that means

(i) The curve $y = f(x)$ is continuous in $[a, b]$, from $A(a, f(a))$ to $B(b, f(b))$.

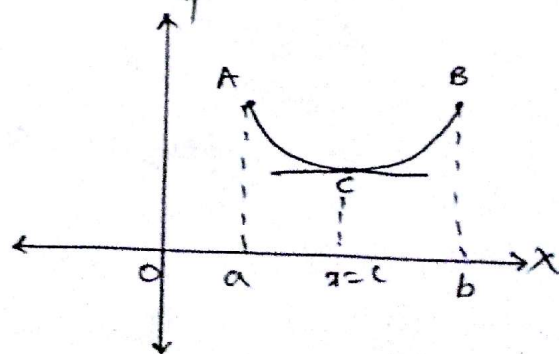
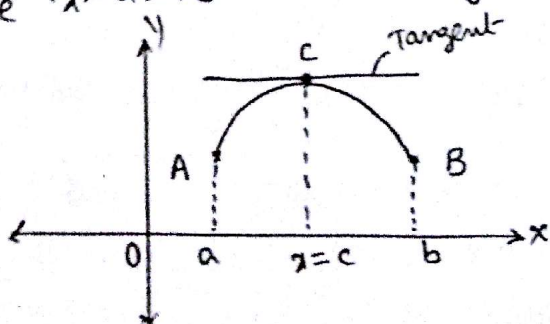
(ii) At every point $x = c$ where $a < c < b$, at the point $C(c, f(c))$ on the curve $y = f(x)$, there is a unique tangent to the curve.

(iii) $f(a) = f(b)$.

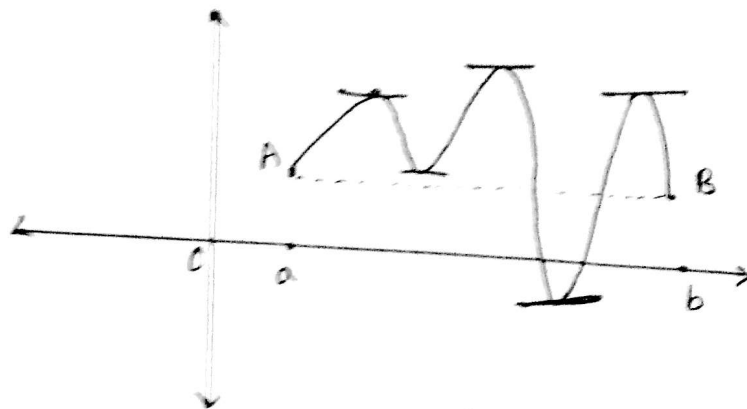
i.e., the two end points of the curve $y = f(x)$ corresponding to $x = a$, $x = b$ have the same ordinates.

i.e., they are at the same height with reference to the x -axis.

So that under three conditions, there is at least one point on the curve where the tangent is parallel to the x -axis according to Rolle's theorem.



In generally, there are many points on the curve $y=f(x)$ where the tangents drawn are all parallel to x -axis.



Note:- Rolle's theorem is not applicable for $f(x)$ if $f(x)$ does not satisfy any condition of Rolle's theorem in the interval.

PROBLEMS:

(1) Verify Rolle's theorem for $f(x) = (x+2)^3(x-3)^4$ in $[-2, 3]$

Sol: Given $f(x) = (x+2)^3(x-3)^4, \forall x \in [-2, 3]$

We know that every polynomial in x is continuous and derivable in $\mathbb{R}$.

$\therefore f(x)$ is polynomial in $x \Rightarrow f(x)$ is continuous in $[-2, 3]$.

$$\begin{aligned} \text{Now } f'(x) &= 3(x+2)^2(x-3)^4 + (x+2)^3 \cdot 4(x-3)^3 \\ &= (x+2)^2(x-3)^3 [3(x-3) + 4(x+2)] \\ &= (x+2)^2(x-3)^3 [7x-1] \end{aligned}$$

Thus $f'(x)$ exists on $(-2, 3)$

$\therefore f(x)$ is derivable on $(-2, 3)$

$$\begin{aligned} \text{and } f(-2) &= (-2+2)^3(-2-3)^4 = 0 \\ f(3) &= (3+2)^3(3-3)^4 = 0 \end{aligned}$$

$$\therefore f(-2) = f(3)$$

So that, Given function $f(x)$ satisfying three conditions of Rolle's theorem in $[-2, 3]$.

Hence $\exists c \in (-2, 3) \ni f'(c) = 0$.

Verification: Consider $f'(c) = 0$
 $(c+2)^2 (c-3)^3 (7c-1) = 0$
 $c = -2, c = 3, c = \frac{1}{7}$

clearly $c = \frac{1}{7} \in (-2, 3)$.

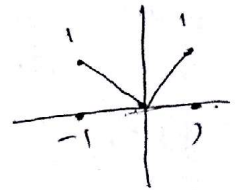
Hence Rolle's theorem is verified.

(2) Discuss the applicability of Rolle's theorem $f(x) = |x|$ in $[-1, 1]$

Sol:

Given $f(x) = |x|, \forall x \in [-1, 1]$

$$= \begin{cases} x, & \text{if } x > 0 \\ -x, & \text{if } x < 0 \end{cases}$$



We have $f(x)$ is continuous for all x

$\therefore f(x)$ is continuous in $[-1, 1]$.

Now, we observed $f(x)$ is not derivable at 0

$$\begin{aligned} \text{i.e., LHD: } Lf'(0) &= \lim_{x \rightarrow 0^-} \frac{f(x) - f(0)}{x - 0} = \lim_{x \rightarrow 0^-} \frac{-x - 0}{x} \\ &= \lim_{x \rightarrow 0^-} (-1) \\ &= -1 \end{aligned}$$

$$\begin{aligned} \text{RAD: } Rf'(0) &= \lim_{x \rightarrow 0^+} \frac{f(x) - f(0)}{x - 0} \\ &= \lim_{x \rightarrow 0^+} \frac{x - 0}{x} \\ &= \lim_{x \rightarrow 0^+} (1) \\ &= 1 \end{aligned}$$

$$\therefore Lf'(0) \neq Rf'(0)$$

$\therefore f(x)$ is not derivable at $x = 0 \in (-1, 1)$

$\therefore f$ is not derivable on $(-1, 1)$.

$\therefore$ Rolle's theorem is not applicable for $f(x)$ on $[-1, 1]$.

(3) Verify Rolle's theorem for $f(x) = x(x+3)e^{-x/2}$ in $[-3, 0]$

Sol: Given $f(x) = x(x+3)e^{-x/2}$, $\forall x \in [-3, 0]$

Since $x(x+3)$ being a polynomial is continuous $\forall x$
and $e^{-x/2}$ is also continuous $\forall x$.

$\therefore f(x) = x(x+3)e^{-x/2}$ is also continuous $\forall x$
in particular $f(x)$ is continuous in $[-3, 0]$

$$\begin{aligned}\text{Now } f'(x) &= x(x+3) \cdot \frac{-1}{2} e^{-x/2} + x \cdot e^{-x/2} + (x+3) e^{-x/2} \\ &= e^{-x/2} \left[-\frac{1}{2}(x^2+3x) + 2x+3 \right] \\ &= e^{-x/2} \left[\frac{6+x-x^2}{2} \right] \\ &= \text{exists } \forall x \in (-3, 0)\end{aligned}$$

$\therefore f(x)$ is derivable in $(-3, 0)$

$$\text{And } f(-3) = 0, \quad f(0) = 0$$

$$\therefore f(-3) = f(0)$$

Thus $f(x)$ satisfies all conditions of Rolle's theorem in $[-3, 0]$.

$$\text{Hence } \exists c \in (-3, 0) \ni f'(c) = 0$$

$$\text{Verification: let } f'(c) = 0 \Rightarrow e^{-c/2} \left[\frac{6+c-c^2}{2} \right] = 0$$

$$e^{-c/2} \neq 0, \text{ for any } c;$$

$$6+c-c^2=0$$

$$c^2-c-6=0$$

$$(c-3)(c+2)=0$$

$$c=-2, c=3$$

$$\text{clearly } \boxed{c=-2} \in (-3, 0)$$

Hence Rolle's theorem verified.

(4) Verify Rolle's theorem for $f(x) = e^x (\sin x - \cos x)$ in $[\frac{\pi}{4}, \frac{5\pi}{4}]$

Sol: Given $f(x) = e^x [\sin x - \cos x]$, $\forall x \in [\frac{\pi}{4}, \frac{5\pi}{4}]$

Since e^x , $\sin x$ and $\cos x$ are continuous $\forall x$
 $\therefore f(x) = e^x [\sin x - \cos x]$ is continuous in $[\frac{\pi}{4}, \frac{5\pi}{4}]$

$$\text{Now } f'(x) = e^x [\cos x + \sin x] + e^x [\sin x - \cos x]$$

$$= e^x [2 \sin x]$$

$$= 2e^x \sin x$$

$$= \text{exists}, \forall x \in (\frac{\pi}{4}, \frac{5\pi}{4})$$

$\therefore f(x)$ is derivable on $(\frac{\pi}{4}, \frac{5\pi}{4})$

$$\text{And } f(\frac{\pi}{4}) = e^{\pi/4} [\sin \frac{\pi}{4} - \cos \frac{\pi}{4}] = e^{\pi/4} [\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}] = 0$$

$$f(\frac{5\pi}{4}) = e^{5\pi/4} [\sin \frac{5\pi}{4} - \cos \frac{5\pi}{4}] = e^{5\pi/4} [-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}] = 0$$

$$\therefore f(\frac{\pi}{4}) = f(\frac{5\pi}{4})$$

Thus given $f(x)$ satisfies all conditions of Rolle's theorem in $[\frac{\pi}{4}, \frac{5\pi}{4}]$

Hence $\exists c \in (\frac{\pi}{4}, \frac{5\pi}{4})$ s.t. $f'(c) = 0$.

Verification: Consider $f'(c) = 0$
 $e^c [2 \sin c] = 0$

$$e^c \neq 0, \text{ for any } c, \quad 2 \sin c = 0$$

$$\sin c = 0 = \sin \pi$$

$$\therefore c = \pi \in (\frac{\pi}{4}, \frac{5\pi}{4})$$

Hence Rolle's theorem is verified.

H.W. Problems:

1) Verify Rolle's theorem for the functions

(i) $f(x) = 2x^3 + x^2 - 4x - 2$ in $[-\sqrt{3}, \sqrt{3}]$

(ii) $f(x) = \frac{1}{x^2}$ in $[-1, 1]$ (not cont. at 0)

(iii) $f(x) = \log \left[\frac{x^2 + ab}{x(a+b)} \right]$ in $[a, b]$, $a > 0, b > 0$

(iv) $f(x) = x^{2/3} - 2x^{1/3}$ in $[0, 8]$

(v) $f(x) = \frac{x^2 - x - 6}{x - 1}$ in $(-2, 3)$

(vi) $f(x) = 1 + x^{2/3}$ for $x \in [-8, 1]$

II. LAGRANGE'S MEAN VALUE THEOREM (or) Mean value Theorem

Statement:- Let $f: [a, b] \rightarrow \mathbb{R}$ be a function such that
i) it is continuous in $[a, b]$ and (ii) it is differentiable in (a, b) . Then there exists at least one point c in (a, b) such that $f'(c) = \frac{f(b) - f(a)}{b - a}$.

* Geometric Interpretation of L.M.T:-

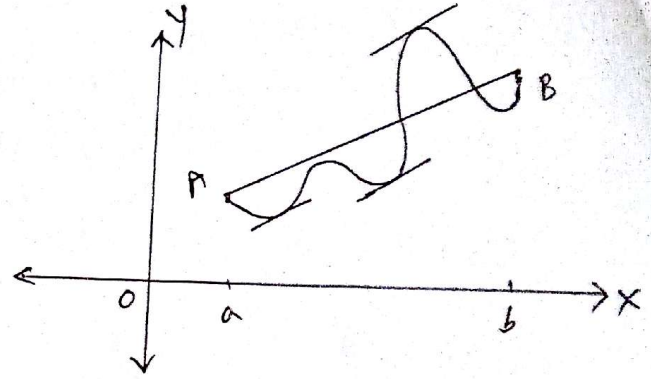
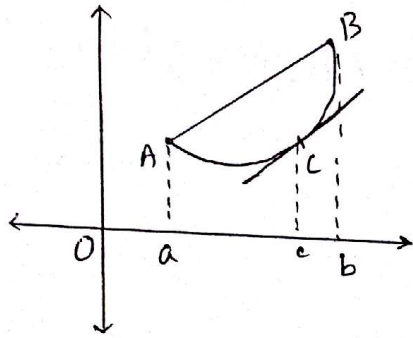
Geometrically, let the function $f: [a, b] \rightarrow \mathbb{R}$ satisfying two conditions of L.M.T. Then

(i) The curve $y = f(x)$ is continuous from $A(a, f(a))$ to $B(b, f(b))$.

(ii) At every point $x = c$, where $a < c < b$, at the point $c(c, f(c))$ on the curve $y = f(x)$, there is a unique tangent to the curve.

Then L.M.T says that there is at least one

point on the curve where the tangent to the curve is parallel to the chord joining the end points A and B on the curve. Since the slope at C, $f'(c)$ is equal to the slope of the chord AB = $\frac{f(b)-f(a)}{b-a}$



Note:- If $f(x)$ is not satisfying any condition of L.M.T. Then L.M.T is not applicable to $f(x)$ in the interval.

PROBLEMS:

- (1) Verify Lagrange's mean value theorem for $f(x) = x^3 - x^2 - 5x + 3$ in $[0, 4]$

Sol: Given $f(x) = x^3 - x^2 - 5x + 3, \forall x \in [0, 4]$
clearly, given function is polynomial

$\therefore f(x)$ is continuous $\forall x$.

In particular, $f(x)$ is continuous in $[0, 4]$

$$\text{Now } f'(x) = 3x^2 - 2x - 5$$

$$= \text{exists}, \forall x \in (0, 4)$$

$\therefore f(x)$ is derivable in $(0, 4)$

Thus, given $f(x)$ satisfies all conditions of L.M.T in $[0, 4]$

hence $\exists c \in (0, 4) \ni f'(c) = \frac{f(4) - f(0)}{4 - 0}$

Verification:- consider $f'(c) = \frac{f(4) - f(0)}{4 - 0}$

$$3c^2 - 2c - 5 = \frac{4^3 - 4^2 - 5(4) + 3 + 0 + 0 + 0 - 3}{4}$$

$$3c^2 - 2c - 5 = \frac{64 - 16 - 20}{4}$$

$$3c^2 - 2c - 5 = \frac{28}{4}$$

$$3c^2 - 2c - 12 = 0$$

$$c = \frac{2 \pm \sqrt{4 + 144}}{6}$$

$$= \frac{2 \pm \sqrt{148}}{6}$$

$$= \frac{2 \pm 2\sqrt{37}}{6}$$

$$c = \frac{1}{3} \pm \frac{\sqrt{37}}{3}$$

We see that $c = \frac{1}{3} + \frac{1}{3}\sqrt{37} \in (0, 4)$

$\left[\because \frac{1}{3} - \frac{1}{3}\sqrt{37} \notin (0, 4) \right]$

$\therefore$ L.M.T is verified.

(2) Verify L.M.T for $f(x) = \log_e x$ in $[1, e]$

Sol:

Given $f(x) = \log_e x$, $\forall x \in [1, e]$

clearly, $f(x)$ is continuous in $[1, e]$

and $f'(x) = \frac{1}{x}$, $\forall x \in (1, e)$
= exists

Thus, given $f(x)$ satisfies all conditions of L.M.T in $[1, e]$

Hence $\exists c \in (1, e)$ $\therefore f'(c) = \frac{f(e) - f(1)}{e - 1}$

verification: consider $f'(c) = \frac{f(e) - f(1)}{e - 1}$

$$\frac{1}{c} = \frac{\log_e e - \log_e 1}{e - 1}$$

$$\frac{1}{c} = \frac{1 - 0}{e - 1}$$

$$\therefore c = e - 1 \in (1, e)$$

$\therefore$ Hence L.M.T Verified.

(3) State whether L.M.T can be applied to the function $f(x) = x^{1/3}$ in $[-1, 1]$? Justify your answer.

Sol:

Given $f(x) = x^{1/3}, \forall x \in [-1, 1]$

Clearly $f(x)$ is continuous in $[-1, 1]$

$$\text{and } f'(x) = \frac{1}{3} x^{-2/3} \\ = \frac{1}{3 x^{2/3}}$$

= does not exist at $x = 0 \in (-1, 1)$

$\therefore f(x)$ is not derivable at $0 \in (-1, 1)$.

$\therefore$ Hence we cannot apply L.M.T to $f(x)$

in $[-1, 1]$.

(4) Give an example of a function that is continuous on $[-1, 1]$ and for which mean value theorem does not hold with explanations.

Sol:

The function $f(x) = |x|$ is continuous in $[-1, 1]$

But it is not derivable at $x = 0 \in (-1, 1)$.

So that L.M.T is not applicable for $f(x)$ in $[-1, 1]$

(5) If $a < b$, prove that $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$ using L.M.T. Deduce that $\frac{\pi}{4} + \frac{3}{25} < \tan^{-1}\frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$

Sol: let the function $f(x) = \tan^{-1}x$, $\forall x \in [a, b]$, $a < b$

Since, clearly $f(x)$ is continuous on $[a, b]$

and $f'(x) = \frac{1}{1+x^2}$, $\forall x \in (a, b)$

$\therefore f(x)$ is also derivable in (a, b)

Thus given $f(x)$ is satisfies all conditions of L.M.T in $[a, b]$.

Hence $\exists c \in (a, b)$ s.t. $f'(c) = \frac{f(b) - f(a)}{b - a}$

$$\Rightarrow \frac{1}{1+c^2} = \frac{\tan^{-1}b - \tan^{-1}a}{b - a} \quad \text{--- (1)}$$

Since $c \in (a, b)$

$$\Rightarrow a < c < b$$

$$\Rightarrow a^2 < c^2 < b^2$$

$$\Rightarrow 1+a^2 < 1+c^2 < 1+b^2$$

$$\Rightarrow \frac{1}{1+a^2} > \frac{1}{1+c^2} > \frac{1}{1+b^2}$$

$$\text{i.e., } \frac{1}{1+b^2} < \frac{1}{1+c^2} < \frac{1}{1+a^2}$$

using eq (1), $\frac{1}{1+b^2} < \frac{\tan^{-1}b - \tan^{-1}a}{b - a} < \frac{1}{1+a^2}$

$$\therefore \frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$$

$\therefore$ Hence the proved, where $a < b$.

Deduction:

We have $\frac{b-a}{1+b^2} < \tan^{-1}b - \tan^{-1}a < \frac{b-a}{1+a^2}$

Taking $b = \frac{4}{3}$, $a = 1$, we get

$$\frac{4/3^{-1}}{1 + \frac{16}{9}} < \tan^{-1} \frac{4}{3} - \tan^{-1}(1) < \frac{4/3^{-1}}{1+1}$$

$$\Rightarrow \frac{3}{25} < \tan^{-1} \frac{4}{3} - \frac{\pi}{4} < \frac{1}{6}$$

$$\therefore \frac{\pi}{4} + \frac{3}{25} < \tan^{-1} \frac{4}{3} < \frac{\pi}{4} + \frac{1}{6}$$

* H.W. PROBLEMS :

(1) Verify L.M.T for the functions

(i) $f(x) = x(x-2)(x-3)$ in $(0,4)$

(ii) $f(x) = (x-1)(x-2)(x-3)$ on $[0,4]$

(2) Show that for any $x > 0$, $1+x < e^x < 1+xe^x$

(3) Prove that $\frac{\pi}{6} + \frac{1}{5\sqrt{3}} < \sin^{-1}(\frac{3}{5}) < \frac{\pi}{6} + \frac{1}{8}$.

* III

CAUCHY'S MEAN VALUE THEOREM :

Statement :- If $f: [a,b] \rightarrow \mathbb{R}$, $g: [a,b] \rightarrow \mathbb{R}$ are such that
 (i) f, g are continuous on $[a,b]$ (ii) f, g are derivable on (a,b)
 and (iii) $g'(x) \neq 0 \forall x \in (a,b)$. Then there exists a point
 $c \in (a,b)$ such that $\frac{f(b)-f(a)}{g(b)-g(a)} = \frac{f'(c)}{g'(c)}$

Note :- Cauchy's mean value theorem is applicable for $f(x)$ and $g(x)$ in $[a,b]$ if both satisfying all conditions of C.M.T in $[a,b]$, otherwise not applicable.

PROBLEMS:

- (1) Find c of Cauchy's mean value theorem for $f(x) = \sqrt{x}$, and $g(x) = \frac{1}{\sqrt{x}}$ in $[a, b]$, where $0 < a < b$

(OR)

If $f(x) = \sqrt{x}$ and $g(x) = \frac{1}{\sqrt{x}}$ prove that 'c' of the Cauchy's generalized mean value theorem is the geometric mean of a and b for any $a > 0, b > 0$

Sol: Given $f(x) = \sqrt{x}$, $g(x) = \frac{1}{\sqrt{x}}$, $\forall x \in [a, b]$, $a < b$
clearly $f(x)$ and $g(x)$ are continuous on $[a, b]$
and derivable on (a, b)
also $g'(x) = \frac{-1}{2x^{3/2}} \neq 0, \forall x \in (a, b)$, $f'(x) = \frac{1}{2\sqrt{x}}$

Thus $f(x)$ & $g(x)$ satisfies all conditions of C.M.T in $[a, b]$.

$$\text{Hence } \exists c \in (a, b) \text{ s.t. } \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$$

$$\therefore \frac{\sqrt{b} - \sqrt{a}}{\frac{1}{\sqrt{b}} - \frac{1}{\sqrt{a}}} = \frac{\frac{1}{2\sqrt{c}}}{-\frac{1}{2c\sqrt{c}}}$$

$$\frac{\sqrt{b} - \sqrt{a}}{-\frac{(\sqrt{b} - \sqrt{a})}{\sqrt{ab}}} = -c$$

$$\boxed{\therefore c = \sqrt{ab}} \in (a, b)$$

i.e., c is geometric mean of a & b .

Hence $c \in (a, b)$ which verifies C.M.T.

(2) If $f(x) = \log x$ and $g(x) = x^2$ in $[a, b]$ with $b > a > 1$, using C.M.T. Prove that $\frac{\log b - \log a}{b - a} = \frac{a+b}{2c^2}$

Sol:- Given $f(x) = \log x$, $g(x) = x^2$, $\forall x \in [a, b]$, $b > a > 1$, clearly, $f(x)$, $g(x)$ are continuous on $[a, b]$ and derivable on (a, b) .

also $f'(x) = \frac{1}{x}$, $g'(x) = 2x \neq 0$, $\forall x \in (a, b)$

$\therefore$ By C.M.T, $\exists c \in (a, b) \ni \frac{f(b) - f(a)}{g(b) - g(a)} = \frac{f'(c)}{g'(c)}$

$$\therefore \frac{\log b - \log a}{b^2 - a^2} = \frac{1/c}{2c}$$

$$\frac{\log b - \log a}{b - a} = \frac{a+b}{2c^2}$$

$\therefore$ — Hence Proved.

H.W. PROBLEMS:

- (1) Verify C.M.T $f(x) = x^2$, $g(x) = x^3$ in $[1, 2]$ $(c = \frac{14}{9})$
- (2) Find 'c' of C.M.T on $[a, b]$ for $f(x) = e^x$ and $g(x) = e^{-x}$ ($a, b > 0$) $(c = \frac{a+b}{2})$
- (3) Discuss the applicability of C.M.T for the functions $f(x) = \frac{1}{x^2}$, $g(x) = \frac{1}{x}$ on $[a, b]$ $(c = \frac{2ab}{a+b})$
- (4) Verify C.M.T for $f(x) = \sin x$ and $g(x) = \cos x$ on $[0, \frac{\pi}{2}]$. $(c = \pi/6)$
- (5) Verify C.M.T for $f(x)$ and $f'(x)$ in $(1, e)$ given $f(x) = \log x$. $(c = \frac{e}{e-1})$

GENERALIZED MEAN VALUE THEOREMS:

* (1) TAYLOR'S THEOREM:

Statement:- If $f: [a, b] \rightarrow \mathbb{R}$ is such that

- (i) $f^{(n-1)}$ is continuous on $[a, b]$ (ii) $f^{(n-1)}$ is derivable on (a, b)
(or) $f^{(n)}$ exists on (a, b) and $p \in \mathbb{Z}^+$, then there exists a point $c \in (a, b)$ such that-

$$f(b) = f(a) + \frac{(b-a)}{1!} f'(a) + \frac{(b-a)^2}{2!} f''(a) + \dots + \frac{(b-a)^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

$$\text{where } R_n = \frac{(b-a)^p (b-c)^{n-p}}{(n-1)! p} f^{(n)}(c).$$

* NOTE:-

- (1) In the Taylor's theorem the remainder

$$R_n = \frac{(b-a)^p (b-c)^{n-p}}{(n-1)! p} f^{(n)}(c) \text{ is called Schlomitch-Roche's}$$

form of Remainder.

- (2) putting $p=n$, we get the remainder

$$R_n = \frac{(b-a)^n}{n!} f^{(n)}(c) \text{ is called Lagrange's form of Remainder}$$

- (3) putting $p=1$, we get remainder

$$R_n = \frac{(b-a)(b-c)^{n-1}}{(n-1)!} f^{(n)}(c) \text{ is called Cauchy's form of Remainder.}$$

* Another form:-

- If $f: [a, a+h] \rightarrow \mathbb{R}$ is such that (i) $f^{(n-1)}$ is continuous on $[a, a+h]$
(ii) $f^{(n-1)}$ is derivable on $(a, a+h)$ and $p \in \mathbb{Z}^+$ then there exists a real no' $\theta \in (0, 1)$ such that

$$f(a+h) = f(a) + \frac{h}{1!} f'(a) + \frac{h^2}{2!} f''(a) + \dots + \frac{h^{n-1}}{(n-1)!} f^{(n-1)}(a) + R_n$$

$$\text{where } R_n = \frac{h^n (1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(a+\theta h).$$

II MACLAURIN'S THEOREM:-

statement: If $f: [0, x] \rightarrow \mathbb{R}$ is such that (i) $f^{(n-1)}$ is continuous on $[0, x]$ (ii) $f^{(n-1)}$ is derivable on $(0, x)$ and $p \in \mathbb{Z}^+$ then there exists a real no' $\theta \in (0, 1)$ such that-

$$f(x) = f(0) + x f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^{n-1}}{(n-1)!} f^{(n-1)}(0) + R_n$$

$$\text{where } R_n = \frac{x^n (1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(\theta x).$$

NOTE:-

(1) In the Maclaurin's theorem the remainder $R_n = \frac{x^n (1-\theta)^{n-p}}{(n-1)! p} f^{(n)}(\theta x)$ is called Schlomilch-Roché's form of Remainder

(2) putting $p=n$, we get remainder $R_n = \frac{x^n}{n!} f^{(n)}(\theta x)$ is called Lagrange's form of Remainder

(3) putting $p=1$, we get remainder $R_n = \frac{x^n (1-\theta)^{n-1}}{(n-1)!} f^{(n)}(\theta x)$ is called Cauchy's form of Remainder.

Note:- The expressions of $f(b)$, $f(a+h)$ and $f(x)$ are finite sums.

However as $n \rightarrow \infty$, Suppose the remainder after n^{th} term tends to zero, then the above expressions turn out to be an infinite series.

* TAYLOR'S SERIES EXPANSION:-

If the function $f(x)$ has successive derivatives of all orders for $x \in [a, b]$ and the remainder after n^{th} term tends to zero as $n \rightarrow \infty$ then

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \dots + \frac{(x-a)^n}{n!}f^{(n)}(a) + \dots$$

is called Taylor's Series expansion of $f(x)$ about $x=a$.

i.e, this is Taylor's series expansion of $f(x)$ in power's of " $x-a$ ".

* MACLAURIN SERIES EXPANSION:-

If the function $f(x)$ has successive derivatives of all orders for $x \in [0, b]$, and the remainder after n^{th} term in Lagrange's form of remainder [i.e, $\frac{x^n}{n!}f^{(n)}(\theta x)$], tends to zero as $n \rightarrow \infty$ then

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \dots + \frac{x^n}{n!}f^{(n)}(0) + \dots$$

is called Maclaurin's series expansion of $f(x)$.

* PROBLEMS:-

(1) obtain the Maclaurin series expansion of e^x .

Sol:- let $f(x) = e^x$

$$\therefore f'(x) = e^x, f''(x) = e^x, f'''(x) = e^x \dots$$

$$\text{and } f(0) = 1, f'(0) = 1, f''(0) = 1, f'''(0) = 1 \dots$$

The Maclaurin series expansion of $f(x)$ is given by

$$f(x) = f(0) + xf'(0) + \frac{x^2}{2!}f''(0) + \frac{x^3}{3!}f'''(0) + \dots$$

$$\therefore e^x = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \dots + \frac{x^n}{n!} + \dots$$

(2) obtain Maclaurin Series expansion of $\sin x$.

Sol: let $f(x) = \sin x$

$$\therefore f'(x) = \cos x, f''(x) = -\sin x, f'''(x) = -\cos x \dots$$

$$\text{and } f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = -1 \dots$$

The Maclaurin Series is given by .

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \dots + \frac{x^n}{n!} f^{(n)}(0) + \dots$$

$$\sin x = x - \frac{x^3}{3!} + \dots$$

(3) Show that $\tan x = x + \frac{1}{3}x^3 + \frac{2}{15}x^5 + \dots$

Sol:- let $f(x) = \tan x$

$$\therefore f'(x) = \sec^2 x, f''(x) = 2 \sec x \sec x \tan x \\ = 2 \sec^2 x \tan x.$$

$$f'''(x) = 4 \sec^2 x \tan^2 x + 2 \sec^4 x \\ \vdots$$

$$\text{and } f(0) = 0, f'(0) = 1, f''(0) = 0, f'''(0) = 2 \dots$$

The Maclaurin series expansion of $f(x)$ is

$$f(x) = f(0) + \frac{x^1}{1!} f'(0) + \frac{x^2}{2!} f''(0) + \frac{x^3}{3!} f'''(0) + \dots$$

$$\tan x = 0 + x \cdot 1 + 0 + \frac{x^3}{3!} \cdot 2 + \dots$$

$$\therefore \tan x = x + \frac{1}{3}x^3 + \dots$$

(*)

(4) obtain the Taylor's Series expansion of $\sin x$ in powers of $x - \frac{\pi}{4}$.

Sol: let $f(x) = \sin x$

We have Taylor's Series expansion of $f(x)$ in powers of $x - a$ is given by.

$$\text{Here } a = \frac{\pi}{4}$$

$$f(x) = f(a) + (x-a)f'(a) + \frac{(x-a)^2}{2!}f''(a) + \frac{(x-a)^3}{3!}f'''(a) + \dots$$

$$\therefore f'(x) = \cos x, \quad f''(x) = -\sin x, \quad f'''(x) = -\cos x \dots$$

$$f\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f'\left(\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad f''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}}, \quad f'''\left(\frac{\pi}{4}\right) = -\frac{1}{\sqrt{2}} \dots$$

$$\therefore \sin x = f\left(\frac{\pi}{4}\right) + (x - \frac{\pi}{4})f'\left(\frac{\pi}{4}\right) + \frac{(x - \frac{\pi}{4})^2}{2!}f''\left(\frac{\pi}{4}\right) + \frac{(x - \frac{\pi}{4})^3}{3!}f'''\left(\frac{\pi}{4}\right) + \dots$$

$$\sin x = \frac{1}{\sqrt{2}} + (x - \frac{\pi}{4})\frac{1}{\sqrt{2}} + \frac{(x - \frac{\pi}{4})^2}{2!}\left(-\frac{1}{\sqrt{2}}\right) + \frac{(x - \frac{\pi}{4})^3}{3!}\left(-\frac{1}{\sqrt{2}}\right) + \dots$$

$$\sin x = \frac{1}{\sqrt{2}} \left[1 + (x - \frac{\pi}{4}) - \frac{(x - \frac{\pi}{4})^2}{2!} - \frac{(x - \frac{\pi}{4})^3}{3!} + \dots \right]$$

(5) obtain the Taylor's series expansion of e^x about $x = -1$

(or)
obtain the Taylor's series expansion of e^x in powers of " $x+1$ "

Sol: let $f(x) = e^x$

$$\text{Now put } x+1 = t$$

$$x = t-1$$

$$\therefore f(x) = e^{t-1} = e^t \cdot e^{-1}$$

$$= \frac{1}{e} \left[1 + \frac{t}{1} + \frac{t^2}{2!} + \dots \right]$$

$$f(x) = \frac{1}{e} \left[1 + \frac{x+1}{1} + \frac{(x+1)^2}{2!} + \dots \right]$$

$$\text{i.e. } e^x = \frac{1}{e} \left[1 + \frac{x+1}{1} + \frac{(x+1)^2}{2!} + \dots \right] \quad \forall x$$

* H.W. PROBLEMS:

- (1) obtain Maclaurin series expansion of $\cos x$.
- (2) obtain " " " " for $f(x) = (1+x)^n$
- (3) obtain " " " " of $f(x) = \log_e(1+x)$
- (4) show that $\sqrt{x} = 1 + \frac{1}{2}(x-1) - \frac{1}{8}(x-1)^2 + \dots$ for $0 < x < 2$

FUNCTIONS OF SEVERAL VARIABLES

We have $f(x, y)$ is a function in two variables x and y . And $f(x, y, z)$ is a function in three variables x and y and z . where (x, y) and (x, y, z) are in regions in the xy plane and xyz -plane respectively.

We note that Taking $z = f(x, y)$ (or) $z = f(x, y, z)$.

* NEIGHBOURHOOD OF A POINT (a, b) :-

A set of points lying within a circle of radius $(r) > 0$ centered at (a, b) is called a neighbourhood of (a, b) surrounded by the circular region

$$D: (x-a)^2 + (y-b)^2 \leq r^2$$

* LIMIT OF A FUNCTION OF TWO VARIABLES:

Let $f(x, y)$ be a function of two variables x and y . The real number ' l ' is said to be 'limit of a function' $f(x, y)$ at (a, b) if for given any +ve number ' ϵ ' there exists a +ve number δ such that $|f(x, y) - l| < \epsilon$ for all points (x, y) whenever $|x-a| < \delta, |y-b| < \delta$.

It is denoted by $f(x, y) \rightarrow l$ as $x \rightarrow a, y \rightarrow b$
(or) $f(x, y) \rightarrow l$ as $(x, y) \rightarrow (a, b)$

also can be write $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = l$.

(or)

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = l$$

And note that $\lim_{(x, y) \rightarrow (a, b)} f(x, y) = l \Rightarrow \lim_{x \rightarrow a} f(x, b) = l = \lim_{y \rightarrow b} f(a, y)$

Example:-

(1) Evaluate $\lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2+y^2+1}$

Sol:-

$$\begin{aligned} \lim_{\substack{x \rightarrow 1 \\ y \rightarrow 2}} \frac{2x^2y}{x^2+y^2+1} &= \lim_{x \rightarrow 1} \left\{ \lim_{y \rightarrow 2} \frac{2x^2y}{x^2+y^2+1} \right\} \\ &= \lim_{x \rightarrow 1} \frac{4x^2}{x^2+5} \\ &= \frac{4}{6} \\ &= \frac{2}{3} \end{aligned}$$

H.W. P

(1) If $f(x,y) = \frac{x-y}{2x+y}$ show that $\lim_{x \rightarrow 0} \left\{ \lim_{y \rightarrow 0} f(x,y) \right\} \neq \lim_{y \rightarrow 0} \left\{ \lim_{x \rightarrow 0} f(x,y) \right\}$.

* CONTINUITY OF A FUNCTION OF TWO VARIABLES AT A POINT:

The function $f(x,y)$ is said to be continuous at a point (a,b) if given any +ve number ϵ , there exists a +ve number δ such that $|f(x,y) - f(a,b)| < \epsilon$ whenever $|x-a| < \delta, |y-b| < \delta$ (or) $(x-a)^2 + (y-b)^2 < \delta^2$.

It is denoted by $f(x,y) \rightarrow f(a,b)$ as $(x,y) \rightarrow (a,b)$

i.e., $\lim_{(x,y) \rightarrow (a,b)} f(x,y) = f(a,b)$.

A function which is not continuous is said to be discontinuous.

(OR)

A function $f(x,y)$ is said to be continuous at the point (a,b) if $\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x,y)$ exist and equal to $f(a,b)$.

If $f(x,y)$ is continuous at all points of a region then it is said to be continuous in that region.

* PROBLEMS:

- (1) Show that the function $f(x, y) = \begin{cases} \frac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$ is continuous at $(0, 0)$.

Sol: Given $f(x, y) = \begin{cases} \frac{x^3 + 2y^3}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

clearly $f(0, 0) = 0$

Now at $(0, 0)$, $(x, y) \rightarrow (0, 0)$ $f(x, y) = \frac{x^3 + 2y^3}{x^2 + y^2}$

let $y = mx$.

$$\begin{aligned} \lim_{\substack{(x, y) \rightarrow (0, 0) \\ y = mx}} f(x, y) &= \lim_{x \rightarrow 0} \frac{x^3 + 2y^3}{x^2 + y^2} \\ &= \lim_{x \rightarrow 0} \frac{x^3 + 2m^3x^3}{x^2 + m^2x^2} \\ &= \lim_{x \rightarrow 0} x \left[\frac{1 + 2m^3}{1 + m^2} \right] \\ &= 0 \\ &= f(0, 0) \end{aligned}$$

$$\begin{aligned} \text{(OR)} \quad \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{x^3 + 2y^3}{x^2 + y^2} &= \lim_{x \rightarrow 0} \frac{x^3 + 2y^3}{x^2 + y^2} \quad \text{(or)} \quad \lim_{\substack{y \rightarrow 0 \\ x \rightarrow 0}} \frac{x^3 + 2y^3}{x^2 + y^2} \\ &= \lim_{y \rightarrow 0} \frac{2y^3}{y^2} = 0 \\ &= f(0, 0) \end{aligned}$$

$\therefore \lim_{(x, y) \rightarrow (0, 0)} f(x, y) = f(0, 0)$

$\therefore$ Given $f(x, y)$ is continuous at $(0, 0)$.

- (2) Verify the continuity of the function $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Sol: Given $f(x, y) = \begin{cases} \frac{xy}{x^2 + y^2}, & (x, y) \neq (0, 0) \\ 0, & (x, y) = (0, 0) \end{cases}$

Now discuss continuity of $f(x, y)$ at $(0, 0)$

Here clearly $f(0, 0) = 0$.

Now, Let us consider the limit of the function for testing the continuity along the line $y = mx$.

$$\begin{aligned} \therefore \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y) &= \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} \frac{xy}{x^2 + y^2} = \lim_{x \rightarrow 0} \frac{mx^2}{x^2 + m^2x^2} = \lim_{x \rightarrow 0} \frac{m}{1 + m^2} \\ &= \frac{m}{1 + m^2} \end{aligned}$$

which is different for the different m selected

$\therefore \lim_{\substack{x \rightarrow 0 \\ y \rightarrow 0}} f(x, y)$ does not exist

Consider, $\lim_{x \rightarrow 0} f(x, 0) = \lim_{x \rightarrow 0} \frac{0}{x^2} = 0 = f(0, 0)$

and $\lim_{y \rightarrow 0} f(0, y) = \lim_{y \rightarrow 0} \frac{0}{-y^2} = 0 = f(0, 0)$

$\therefore f(x, y)$ is continuous for given values of x and y

But $f(x, y)$ is not continuous at $(0, 0)$.

* H.W.P:-

(1) show that the function $f(x, y) = \frac{xy}{\sqrt{x^2 + y^2}}, (x, y) \neq (0, 0)$
 $= 0, (x, y) = (0, 0)$

is continuous at the origin.

(2) Investigate the continuity at $(0, 0)$ of $f(x, y) = \frac{x^2 - y^2}{x^2 + y^2}, (x, y) \neq (0, 0)$
 $= 0, (x, y) = (0, 0)$

* LIMIT OF A FUNCTION:-

The function $f(x, y)$ is said to tends to the limit as $x \rightarrow a, y \rightarrow b \iff$ if the limit l is independent of the path followed by the ~~path~~ point (x, y) as $x \rightarrow a$ & $y \rightarrow b$ and we write

$$\lim_{\substack{x \rightarrow a \\ y \rightarrow b}} f(x, y) = l$$

* The function is may (or) may not defined at that point, for that limit to exists.

* PARTIAL DIFFERENTIATION:

(1) let $z = f(x, y)$ be a function of two variables x and y .

Then, if $\lim_{\Delta x \rightarrow 0} \frac{f(x + \Delta x, y) - f(x, y)}{\Delta x}$ exists, then it is

said to be partial derivative (or) partial differential coefficient of z (or) $f(x, y)$ w.r.t x where y is kept constant.

It is denoted by the symbol $\frac{\partial z}{\partial x}$ (or) $\frac{\partial f}{\partial x}$ (or) f_x .

(2) similarly, we can define the partial derivative of $z = f(x, y)$ w.r.t y where x is kept constant, as $\lim_{\Delta y \rightarrow 0} \frac{f(x, y + \Delta y) - f(x, y)}{\Delta y}$.

It is denoted by the symbol $\frac{\partial z}{\partial y}$ (or) $\frac{\partial f}{\partial y}$ (or) f_y .

In generally, the partial derivative of $z = f(x_1, x_2, \dots, x_i, \dots, x_n)$ with respect to x_i keeping other variables constant can be defined by

$$\frac{\partial z}{\partial x_i} = \lim_{\Delta x_i \rightarrow 0} \frac{f(x_1, x_2, \dots, x_i + \Delta x_i, \dots, x_n) - f(x_1, x_2, \dots, x_i, \dots, x_n)}{\Delta x_i} \quad (i = 1, 2, \dots, n)$$

* HIGHER ORDER PARTIAL DERIVATIVES:

We know that $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ are first partial derivative of $f(x, y)$. And also they can be differentiated repeatedly to get higher order partial derivatives.

$$\rightarrow \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial x^2}, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial y^2}, \quad \frac{\partial}{\partial x} \left(\frac{\partial f}{\partial y} \right) = \frac{\partial^2 f}{\partial x \partial y}, \quad \frac{\partial}{\partial y} \left(\frac{\partial f}{\partial x} \right) = \frac{\partial^2 f}{\partial y \partial x} \quad (2)$$

$$\rightarrow \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial x^2} \right) = \frac{\partial^3 f}{\partial x^3}, \quad \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial y^2} \right) = \frac{\partial^3 f}{\partial y^3}, \quad \frac{\partial}{\partial x} \left(\frac{\partial^2 f}{\partial y^2} \right) = \frac{\partial^3 f}{\partial x \partial y^2}, \quad \frac{\partial}{\partial y} \left(\frac{\partial^2 f}{\partial x^2} \right) = \frac{\partial^3 f}{\partial y \partial x^2}$$

and so on.

Note:- 1) $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$, for any $f(x, y)$ have continuous partial derivative

$$(2) \quad \frac{\partial^2 f}{\partial x \partial y} = f_{xy}, \quad \frac{\partial^2 f}{\partial y \partial x} = f_{yx}, \quad \frac{\partial^2 f}{\partial x^2} = f_{xx}, \quad \frac{\partial^2 f}{\partial y^2} = f_{yy} \dots$$

$$(3) \quad \frac{\partial^3 f}{\partial x \partial y \partial z} = f_{xyz}, \quad \frac{\partial^3 f}{\partial x \partial z \partial y} = f_{xzy}, \quad \frac{\partial^3 f}{\partial y \partial z \partial x} = f_{yzx} \dots$$

PROBLEMS:-

- (1) Find first and second order partial derivatives of $ax^2 + 2hxy + by^2$ and verify that $\frac{\partial^2 f}{\partial x \partial y} = \frac{\partial^2 f}{\partial y \partial x}$.

sol:

$$\text{let } f(x, y) = ax^2 + 2hxy + by^2$$

$$\therefore \frac{\partial f}{\partial x} = \frac{\partial}{\partial x} [ax^2 + 2hxy + by^2]$$

$$= 2ax + 2hy \quad (\because y \text{ is constant})$$

$$\text{and } \frac{\partial^2 f}{\partial x^2} = 2a$$

$$\text{Now } \frac{\partial f}{\partial y} = \frac{\partial}{\partial y} (ax^2 + 2hxy + by^2)$$

$$= 2hx + 2by \quad (\because x \text{ is constant})$$

$$\Rightarrow \frac{\partial^2 f}{\partial y^2} = 2b.$$

$$\text{and } \frac{\partial^2 f}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial f}{\partial y} \right] = \frac{\partial}{\partial x} [2hx + 2by] = 2h.$$

$$\frac{\partial^2 f}{\partial y \partial x} = \frac{\partial}{\partial y} \left[\frac{\partial f}{\partial x} \right] = \frac{\partial}{\partial y} [2ax + 2hy] = 2h.$$

$$\therefore \frac{\partial^2 f}{\partial y \partial x} = \frac{\partial^2 f}{\partial x \partial y}$$

(2) If $U = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$ then $\frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0$.

Sol:

Given $U = \frac{1}{\sqrt{x^2+y^2+z^2}}$, $x^2+y^2+z^2 \neq 0$

$$\therefore \frac{\partial U}{\partial x} = \frac{-1}{2\sqrt{x^2+y^2+z^2}} \cdot \frac{\partial}{\partial x} (x^2+y^2+z^2) \quad \left[\because \frac{d}{dx} \frac{1}{\sqrt{x}} = -\frac{1}{2} \frac{1}{x^{3/2}} \right]$$

$$= \frac{-2x}{2\sqrt{x^2+y^2+z^2}^{3/2}}$$

$$= \frac{-x}{\sqrt{x^2+y^2+z^2}^{3/2}}$$

$$\frac{\partial U}{\partial x} = -x (x^2+y^2+z^2)^{-3/2}$$

$$\therefore \frac{\partial^2 U}{\partial x^2} = -x \cdot \frac{-3}{2} (x^2+y^2+z^2)^{-5/2} \cdot (2x) + (x^2+y^2+z^2)^{-3/2}$$

$$= (x^2+y^2+z^2)^{-5/2} [3x^2 - (x^2+y^2+z^2)]$$

$$\frac{\partial^2 U}{\partial x^2} = (x^2+y^2+z^2)^{-5/2} [2x^2 - y^2 - z^2] \quad \text{--- ①}$$

Similarly, we get-

$$\frac{\partial^2 U}{\partial y^2} = (x^2+y^2+z^2)^{-5/2} [2y^2 - x^2 - z^2] \quad \text{--- ②}$$

$$\text{and } \frac{\partial^2 U}{\partial z^2} = (x^2+y^2+z^2)^{-5/2} [2z^2 - x^2 - y^2] \quad \text{--- ③}$$

adding ①, ②, ③

$$\therefore \frac{\partial^2 U}{\partial x^2} + \frac{\partial^2 U}{\partial y^2} + \frac{\partial^2 U}{\partial z^2} = 0.$$

(3) Find $\frac{\partial^3 u}{\partial x \partial y \partial z}$ if $u = e^{xyz}$.

Sol:

Given $u = e^{xyz}$

$$\frac{\partial u}{\partial x} = e^{xyz} \cdot yz, \quad \frac{\partial u}{\partial y} = e^{xyz} \cdot xz, \quad \frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

$$\frac{\partial u}{\partial z} = e^{xyz} \cdot xy$$

$$\begin{aligned}
 \therefore \frac{\partial^3 u}{\partial x \partial y \partial z} &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left[\frac{\partial u}{\partial z} \right] \right] \\
 &= \frac{\partial}{\partial x} \left[\frac{\partial}{\partial y} \left[x y e^{xyz} \right] \right] \\
 &= \frac{\partial}{\partial x} \left[x^2 y z e^{xyz} + x e^{xyz} \right] \\
 &= x^2 y z^2 e^{xyz} + 2 x y z e^{xyz} + x y z e^{xyz} + e^{xyz} \\
 \therefore \frac{\partial^3 u}{\partial x \partial y \partial z} &= e^{xyz} [x^2 y z^2 + 3 x y z + 1]
 \end{aligned}$$

(4) Verify $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$ for the function $u = \tan^{-1} \frac{x}{y}$.

Sol:

Given $u = \tan^{-1} \frac{x}{y}$

$$\frac{\partial u}{\partial x} = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} = \frac{y}{y^2 + x^2}$$

$$\text{and } \frac{\partial u}{\partial y} = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{-x}{y^2} = \frac{-x}{y^2 + x^2}$$

Now

$$\begin{aligned}
 \frac{\partial^2 u}{\partial x \partial y} &= \frac{\partial}{\partial x} \left[\frac{\partial u}{\partial y} \right] = \frac{\partial}{\partial x} \left[\frac{-x}{y^2 + x^2} \right] = \frac{(y^2 + x^2)(-1) + x(2x)}{(y^2 + x^2)^2} \\
 &= \frac{-y^2 - x^2 + 2x^2}{(y^2 + x^2)^2} = \frac{x^2 - y^2}{(y^2 + x^2)^2}
 \end{aligned}$$

$$\begin{aligned}
 \text{and } \frac{\partial^2 u}{\partial y \partial x} &= \frac{\partial}{\partial y} \left[\frac{\partial u}{\partial x} \right] = \frac{\partial}{\partial y} \left[\frac{y}{y^2 + x^2} \right] = \frac{(y^2 + x^2)(1) - y(2y)}{(y^2 + x^2)^2} \\
 &= \frac{x^2 - y^2}{(y^2 + x^2)^2}
 \end{aligned}$$

$$\therefore \frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$$

H.w.P.

(5) (i) If $u = \sin^{-1} \frac{x}{y} + \tan^{-1} \frac{y}{x}$ then Prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = 0$.

Sol: (ii) If $x = x^2 \tan^{-1} \left(\frac{y}{x} \right) - y^2 \tan^{-1} \left(\frac{x}{y} \right)$, then Find value of $x^2 \frac{\partial^2 z}{\partial x^2} + 2xy \frac{\partial^2 z}{\partial x \partial y} + y^2 \frac{\partial^2 z}{\partial y^2} = 24$

(5) If $x^y z^z = e$ Show that at $x=y=z$, $\frac{\partial^2 z}{\partial x \partial y} = - (x \log e x)^{-1}$.

Sol:

Given $x^y z^z = e$

Taking logarithm on B.S

$$x \log x + y \log y + z \log z = \log e = 1$$

$$z \log z = 1 - x \log x - y \log y$$

Diffn partially w.r.t x :

$$z \cdot \frac{1}{z} \frac{\partial z}{\partial x} + \frac{\partial z}{\partial x} \cdot \log z = -x \cdot \frac{1}{x} - \log x - 0$$

$$(1 + \log z) \frac{\partial z}{\partial x} = - [1 + \log x]$$

$$\therefore \frac{\partial z}{\partial x} = - \frac{[1 + \log x]}{1 + \log z}$$

$$\text{at } x=y=z, \quad \frac{\partial z}{\partial x} = - \frac{[\log x + 1]}{1 + \log x}$$

$$\therefore \frac{\partial z}{\partial x} = -1$$

and lly we get, $\frac{\partial z}{\partial y} = - \frac{(1 + \log y)}{1 + \log z}$

$$\text{at } x=y=z, \quad \frac{\partial z}{\partial y} = -1.$$

$$\text{Now } \frac{\partial^2 z}{\partial x \partial y} = \frac{\partial}{\partial x} \left[\frac{\partial z}{\partial y} \right] = \frac{\partial}{\partial x} \left[- \frac{(1 + \log y)}{(1 + \log z)} \right]$$

$$= - (1 + \log y) \cdot \frac{-1}{(1 + \log z)^2} \cdot \frac{1}{z} \cdot \frac{\partial z}{\partial x}$$

$$= \frac{1 + \log y}{(1 + \log z)^2} \cdot \frac{1}{x} \cdot (-1)$$

$$= \frac{1 + \log x}{x (1 + \log x)^2} (-1)$$

$$= \frac{-1}{x (\log e + \log x)}$$

$$\therefore \frac{\partial^2 z}{\partial x \partial y} = - (x \log e x)^{-1}$$

* H.W.P :-

- (1) Find first and second order partial derivatives of
 (i) $\log(x^2+y^2)$ (ii) x^3+y^3-3axy .
- (2) If $u = \log(x^2+y^2+z^2)$, Prove that $(x^2+y^2+z^2) \left[\frac{\partial^2 u}{\partial x^2} + \frac{\partial^2 u}{\partial y^2} + \frac{\partial^2 u}{\partial z^2} \right] = 2$.
- (3) Verify that $\frac{\partial^2 u}{\partial x \partial y} = \frac{\partial^2 u}{\partial y \partial x}$, if $u = \sin^{-1}\left(\frac{x}{y}\right)$
- (4) If $f(x,y) = \log(x^2+y^2) + \tan^{-1}\left(\frac{y}{x}\right)$, then show that $f_{xx} + f_{yy} = 0$.
- * (5) If $u = \frac{y}{z} + \frac{z}{x}$ then prove that $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$.

* COMPOSITE FUNCTION :-

If $z = f(u,v)$, where $u = \phi_1(x,y)$, $v = \phi_2(x,y)$, then z is called composite function of two variables x & y .

* THE CHAIN RULE OF PARTIAL DIFFERENTIATION :-

- (I) let $z = f(u,v)$, where $u = \phi_1(x,y)$, $v = \phi_2(x,y)$. then

$$\frac{\partial z}{\partial x} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial x}$$

$$\frac{\partial z}{\partial y} = \frac{\partial z}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial z}{\partial v} \cdot \frac{\partial v}{\partial y}$$

- (II) let $u = f(u,v,w)$, where $u = \phi_1(x,y,z)$, $v = \phi_2(x,y,z)$,
 $w = \phi_3(x,y,z)$

then

$$\frac{\partial u}{\partial x} = \frac{\partial u}{\partial u} \cdot \frac{\partial u}{\partial x} + \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial x} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial x}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial u} \cdot \frac{\partial u}{\partial y} + \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial y} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial y}$$

$$\frac{\partial u}{\partial z} = \frac{\partial u}{\partial u} \cdot \frac{\partial u}{\partial z} + \frac{\partial u}{\partial v} \cdot \frac{\partial v}{\partial z} + \frac{\partial u}{\partial w} \cdot \frac{\partial w}{\partial z}$$

* EULER'S THEOREM:- If 'u' be a homogeneous fn of degree 'n' in x and y. Then $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} = nu$.
 $x^2 f_{xx} + 2xy f_{xy} + y^2 f_{yy} = n(n-1)u$

(2) If $z = f(x, y)$, $x = e^{2u} + e^{-2v}$; $y = e^{2u} - e^{2v}$ then find $\frac{\partial f}{\partial u} - \frac{\partial f}{\partial v}$.

Sol:
Given $z = f(x, y)$, where $x = e^{2u} + e^{-2v}$
 $y = e^{2u} - e^{2v}$

using chain rule of partial differentiation

$$\therefore \frac{\partial f}{\partial u} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial u} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial u}$$

$$= \frac{\partial f}{\partial x} \cdot 2e^{2u} + \frac{\partial f}{\partial y} \cdot 2e^{2u}$$

$$\text{and } \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} \cdot \frac{\partial x}{\partial v} + \frac{\partial f}{\partial y} \cdot \frac{\partial y}{\partial v}$$

$$= \frac{\partial f}{\partial x} (-2e^{-2v}) + \frac{\partial f}{\partial y} \cdot 2e^{2v}$$

$$\therefore \frac{\partial f}{\partial u} - \frac{\partial f}{\partial v} = \frac{\partial f}{\partial x} (2e^{2u} + 2e^{-2v}) + \frac{\partial f}{\partial y} (2e^{2u} - 2e^{2v})$$

$$= 2(e^{2u} + e^{-2v}) \frac{\partial f}{\partial x} + 2(e^{2u} - e^{2v}) \frac{\partial f}{\partial y}$$

$$= 2x \frac{\partial f}{\partial x} + 2(e^{2u} - e^{2v}) \frac{\partial f}{\partial y}$$

(3) If $f(x, y) = x \cos y + e^x \sin y$; $x = t^2 + 1$, $y = t^3 + t$ find $\frac{df}{dt}$ at $t=0$.

Sol:
Given $f(x, y) = x \cos y + e^x \sin y$, where $x = t^2 + 1$
 $y = t^3 + t$

$\therefore$ Total differential coefficient-

$$\frac{df}{dt} = \frac{\partial f}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial f}{\partial y} \cdot \frac{dy}{dt}$$

$$\text{Now } \frac{\partial f}{\partial x} = \cos y + e^x \sin y, \quad \frac{dx}{dt} = 2t, \quad \frac{dy}{dt} = 3t^2 + 1$$

$$\frac{\partial f}{\partial y} = -x \sin y + e^x \cos y$$

* TOTAL DIFFERENTIAL COEFFICIENT:

Let $z = f(x, y)$, where $x = \phi_1(t)$, $y = \phi_2(t)$

When substituting x and y in $z = f(x, y)$, z becomes a function of a single variable t

i.e. $z = f(\phi_1(t), \phi_2(t))$.

Then the derivative of z w.r.t ' t ', i.e. " $\frac{dz}{dt}$ " is called the total differential coefficient (or) total derivative of z .

$$\therefore \frac{dz}{dt} = \frac{\partial z}{\partial x} \cdot \frac{dx}{dt} + \frac{\partial z}{\partial y} \cdot \frac{dy}{dt}$$

* If $u = f(x, y)$, Then in the differential form the above result can be written as

$du = \frac{\partial u}{\partial x} dx + \frac{\partial u}{\partial y} dy$ is called the total differential of u .

PROBLEMS:-

(1) Find Total differential of the function $z = \tan^{-1}\left(\frac{x}{y}\right)$, $(x, y) \neq (0, 0)$

Sol: Given $z = \tan^{-1}\left(\frac{x}{y}\right)$, $(x, y) \neq (0, 0)$

$$\therefore \text{Total differential } dz = \frac{\partial z}{\partial x} dx + \frac{\partial z}{\partial y} dy$$

$$\therefore dz = \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{1}{y} \cdot dx + \frac{1}{1 + \frac{x^2}{y^2}} \cdot \frac{-x}{y^2} \cdot dy$$

$$= \frac{y}{y^2 + x^2} dx + \frac{-x}{y^2 + x^2} dy$$

$$\therefore dz = \frac{1}{x^2 + y^2} [y dx - x dy]$$

$$\therefore \frac{df}{dt} = (\cos y + e^x \sin y) \cdot at + (-2 \sin y + e^x \cos y) (3t^2 + 1)$$

$$\text{at } t=0, \quad \frac{df}{dt} = 0 + (-2 \sin y + e^x \cos y) (0+1)$$

$$\therefore \frac{df}{dt} = -2 \sin y + e^x \cos y$$

(H) If $u = f(r, s, t)$ where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$, S.T. $x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = 0$

Sol: Given $u = f(r, s, t)$, where $r = \frac{x}{y}$, $s = \frac{y}{z}$, $t = \frac{z}{x}$

Using chain rule,

$$\text{Now } \frac{\partial r}{\partial x} = \frac{1}{y}, \quad \frac{\partial r}{\partial y} = -\frac{x}{y^2}, \quad \frac{\partial r}{\partial z} = 0$$

$$\frac{\partial s}{\partial x} = 0, \quad \frac{\partial s}{\partial y} = \frac{1}{z}, \quad \frac{\partial s}{\partial z} = -\frac{y}{z^2}$$

$$\frac{\partial t}{\partial x} = -\frac{z}{x^2}, \quad \frac{\partial t}{\partial y} = 0, \quad \frac{\partial t}{\partial z} = \frac{1}{x}$$

$$\therefore \frac{\partial u}{\partial x} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial x} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial x} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial x}$$

$$= \frac{\partial u}{\partial r} \cdot \frac{1}{y} + \frac{\partial u}{\partial s} \cdot 0 + \frac{\partial u}{\partial t} \cdot \left(-\frac{z}{x^2}\right)$$

$$= \frac{1}{y} \frac{\partial u}{\partial r} - \frac{z}{x^2} \frac{\partial u}{\partial t}$$

$$\frac{\partial u}{\partial y} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial y} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial y} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial y}$$

$$= \frac{\partial u}{\partial r} \cdot \left(-\frac{x}{y^2}\right) + \frac{\partial u}{\partial s} \cdot \frac{1}{z} + \frac{\partial u}{\partial t} \cdot 0$$

$$= -\frac{x}{y^2} \frac{\partial u}{\partial r} + \frac{1}{z} \frac{\partial u}{\partial s}$$

$$\text{and } \frac{\partial u}{\partial z} = \frac{\partial u}{\partial r} \cdot \frac{\partial r}{\partial z} + \frac{\partial u}{\partial s} \cdot \frac{\partial s}{\partial z} + \frac{\partial u}{\partial t} \cdot \frac{\partial t}{\partial z}$$

$$= \frac{\partial u}{\partial r} (0) + \frac{\partial u}{\partial s} \cdot \frac{-y}{z^2} + \frac{\partial u}{\partial t} \cdot \frac{1}{z}$$

$$= -\frac{y}{z^2} \frac{\partial u}{\partial s} + \frac{1}{z} \frac{\partial u}{\partial t}$$

$$\therefore x \frac{\partial u}{\partial x} + y \frac{\partial u}{\partial y} + z \frac{\partial u}{\partial z} = x \left[\frac{1}{y} \frac{\partial u}{\partial r} - \frac{z}{z^2} \frac{\partial u}{\partial t} \right] +$$

$$y \left[-\frac{x}{y^2} \frac{\partial u}{\partial r} + \frac{1}{z} \frac{\partial u}{\partial s} \right] +$$

$$z \left[-\frac{y}{z^2} \frac{\partial u}{\partial s} + \frac{1}{z} \frac{\partial u}{\partial t} \right]$$

$$= 0$$

(5) If $u = x \log xy$, where $x^3 + y^3 + 3xy = 1$. find $\frac{du}{dx}$

Sol:

Given $u = x \log xy$ and $x^3 + y^3 + 3xy = 1$.

$$u = x [\log x + \log y]$$

$$u = x \log x + x \log y$$

$$\therefore \frac{\partial u}{\partial x} = x \cdot \frac{1}{x} + \log x + \log y = 1 + \log x + \log y$$

$$\text{and } \frac{\partial u}{\partial y} = \frac{x}{y}$$

$$\text{also } x^3 + y^3 + 3xy = 1$$

Differentiating w.r.t. x

$$3x^2 + 3y^2 \frac{dy}{dx} + 3[x \frac{dy}{dx} + y] = 0$$

$$x^2 + (x + y^2) \frac{dy}{dx} + y = 0$$

$$\frac{dy}{dx} = -\frac{(y + x^2)}{x + y^2}$$

$$\therefore \text{T.D.C } \frac{du}{dx} = \frac{\partial u}{\partial x} \frac{dx}{dx} + \frac{\partial u}{\partial y} \frac{dy}{dx}$$

$$= 1 + \log x + \log y + \frac{x}{y} \cdot \left(\frac{y + x^2}{x + y^2} \right)$$

$$\therefore \frac{du}{dx} = 1 + \log xy - \frac{x}{y} \left(\frac{x^2 + y}{x + y^2} \right)$$

H.W.P:-

- 1) If $z = f(x, y)$ where $x = e^u + e^{-v}$, $y = e^u - e^{-v}$, s.t. $\frac{\partial z}{\partial u} - \frac{\partial z}{\partial v} = x \frac{\partial z}{\partial x} - y \frac{\partial z}{\partial y}$.
- 2) If $u = f(y-z, z-x, x-y)$, Prove that $\frac{\partial u}{\partial x} + \frac{\partial u}{\partial y} + \frac{\partial u}{\partial z} = 0$.
- 3) If $u = x^2 y^3$, where $x = \log t$ and $y = e^t$ find $\frac{du}{dt}$.

* JACOBIAN :-

Let $u = u(x, y)$, $v = v(x, y)$ be two functions in x and y .

The determinant $\begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$ (or) $\begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix}$

is called the Jacobian of u and v w.r.t x, y (or) the Jacobian of the transformation.

It is denoted by $\frac{\partial(u, v)}{\partial(x, y)}$ (or) $J\left(\frac{u, v}{x, y}\right)$

i.e., $\frac{\partial(u, v)}{\partial(x, y)}$ (or) $J\left(\frac{u, v}{x, y}\right) = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix}$

* NOTE:- If $u = u(x, y, z)$, $v = v(x, y, z)$, $w = w(x, y, z)$, then

the Jacobian of u, v, w w.r.t x, y, z is given by

$$\frac{\partial(u, v, w)}{\partial(x, y, z)} = \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} & \frac{\partial u}{\partial z} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} & \frac{\partial v}{\partial z} \\ \frac{\partial w}{\partial x} & \frac{\partial w}{\partial y} & \frac{\partial w}{\partial z} \end{vmatrix} \quad \text{(or)} \quad \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix}$$

- 2) Similarly we can define Jacobian of n indep. variables.
- 3) This Jacobian of transformation is useful particularly when we transform the variables conveniently while performing multiple integration.

An imp. application of Jacobian is in connection with the change of variables in multiple integrals.

* PROPERTIES OF JACOBIANS:

$$(1) \frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1.$$

(2) If u, v are functions of x, y and x, y are functions of u, v . i.e., $u = u(x, y)$
 $v = v(x, y)$ and $x = x(u, v)$
 $y = y(u, v)$

$$\text{then } \frac{\partial(u,v)}{\partial(x,y)} = \frac{\partial(u,v)}{\partial(x,y)} \cdot \frac{\partial(x,y)}{\partial(x,y)}$$

* PROBLEMS :-

(1) If $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$ show that $\frac{\partial(u,v,w)}{\partial(x,y,z)} = 4$.

Sol: Given $u = \frac{yz}{x}$, $v = \frac{zx}{y}$, $w = \frac{xy}{z}$

$$\therefore \frac{\partial u}{\partial x} = -\frac{yz}{x^2}, \quad \frac{\partial v}{\partial x} = \frac{z}{y}, \quad \frac{\partial w}{\partial x} = \frac{y}{z}$$

$$\frac{\partial u}{\partial y} = \frac{z}{x}, \quad \frac{\partial v}{\partial y} = -\frac{zx}{y^2}, \quad \frac{\partial w}{\partial y} = \frac{x}{z}$$

$$\frac{\partial u}{\partial z} = \frac{y}{x}, \quad \frac{\partial v}{\partial z} = \frac{x}{y}, \quad \frac{\partial w}{\partial z} = -\frac{xy}{z^2}$$

$$\therefore \frac{\partial(u,v,w)}{\partial(x,y,z)} = \begin{vmatrix} u_x & u_y & u_z \\ v_x & v_y & v_z \\ w_x & w_y & w_z \end{vmatrix} = \begin{vmatrix} -\frac{yz}{x^2} & \frac{z}{y} & \frac{y}{z} \\ \frac{z}{x} & -\frac{zx}{y^2} & \frac{x}{z} \\ \frac{y}{x} & \frac{x}{y} & -\frac{xy}{z^2} \end{vmatrix}$$

$$= \frac{1}{x^2} \cdot \frac{1}{y^2} \cdot \frac{1}{z^2} \begin{vmatrix} -yz & z^2 & y^2 \\ yz & -zx^2 & xy \\ yz & x^2 & -xy \end{vmatrix}$$

$$= \frac{1}{(xyz)^2} \begin{vmatrix} -yz & z^2 & y^2 \\ yz & -zx^2 & xy \\ yz & x^2 & -xy \end{vmatrix}$$

$$= -1[2-1] - 1[-1-1] + 1[1+1]$$

$$= -0 + 2 + 2$$

$$= 4$$

(2) If $x = u(1+v)$, $y = v(1+u)$ then p.T $\frac{\partial(x,y)}{\partial(u,v)} = 1+u+v$.

SA: Given $x = u(1+v)$, $y = v(1+u)$

$$\frac{\partial x}{\partial u} = 1+v, \quad \frac{\partial y}{\partial u} = v$$
$$\frac{\partial x}{\partial v} = u, \quad \frac{\partial y}{\partial v} = 1+u$$

$$\therefore \frac{\partial(x,y)}{\partial(u,v)} = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} 1+v & u \\ v & 1+u \end{vmatrix} = (1+v)(1+u) - uv$$
$$= 1+u+v+uv - uv$$
$$= 1+u+v$$

(3) If $x = uv$, $y = \frac{u}{v}$ then Prove that $J\left[\frac{x,y}{u,v}\right] \cdot J'\left[\frac{u,v}{x,y}\right] = 1$.

SA: Given $x = uv$, $y = \frac{u}{v}$

now solving for u and v , we get u and v are in terms of x and y .

$$\Rightarrow u = \frac{x}{v}, \quad u = yv$$
$$\therefore \frac{x}{v} = yv \Rightarrow v = \sqrt{\frac{x}{y}}$$

$$u_y, \quad u = \sqrt{xy}$$

$$\text{Now } J\left[\frac{x,y}{u,v}\right] = \begin{vmatrix} x_u & x_v \\ y_u & y_v \end{vmatrix} = \begin{vmatrix} v & u \\ \frac{1}{v} & -\frac{u}{v^2} \end{vmatrix} = -\frac{u}{v} - \frac{u}{v} = -\frac{2u}{v} = -2y$$

$$\text{and } J'\left[\frac{u,v}{x,y}\right] = \begin{vmatrix} u_x & u_y \\ v_x & v_y \end{vmatrix} = \begin{vmatrix} \frac{\sqrt{y}}{2\sqrt{x}} & \frac{\sqrt{x}}{2\sqrt{y}} \\ \frac{1}{2\sqrt{x}\sqrt{y}} & \frac{-\sqrt{x}}{2y\sqrt{y}} \end{vmatrix}$$
$$= -\frac{1}{2y} - \frac{1}{2y} = -\frac{1}{y}$$

$$\therefore J\left[\frac{x,y}{u,v}\right] \cdot J'\left[\frac{u,v}{x,y}\right] = -\frac{2y}{v} \cdot -\frac{1}{2y} = -2y \cdot -\frac{1}{2y} = 1$$

(4) If $x = r \cos \theta$, $y = r \sin \theta$, find $\frac{\partial(x, y)}{\partial(r, \theta)}$.

Sol: Given $x = r \cos \theta$, $y = r \sin \theta$

$$\frac{\partial x}{\partial r} = \cos \theta$$

$$\frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial x}{\partial \theta} = -r \sin \theta$$

$$\frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\begin{aligned} \therefore \frac{\partial(x, y)}{\partial(r, \theta)} &= \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} = \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= r \cos^2 \theta + r \sin^2 \theta \\ &= r \end{aligned}$$

(5) If $u = x^2 - y^2$, $v = 2xy$ and $x = r \cos \theta$, $y = r \sin \theta$, find $\frac{\partial(u, v)}{\partial(r, \theta)}$.

Sol: Given $u = x^2 - y^2$, $v = 2xy$ and $x = r \cos \theta$, $y = r \sin \theta$

$$\frac{\partial u}{\partial x} = 2x, \quad \frac{\partial v}{\partial x} = 2y \quad \text{and} \quad \frac{\partial x}{\partial r} = \cos \theta, \quad \frac{\partial y}{\partial r} = \sin \theta$$

$$\frac{\partial u}{\partial y} = -2y, \quad \frac{\partial v}{\partial y} = 2x \quad \frac{\partial x}{\partial \theta} = -r \sin \theta, \quad \frac{\partial y}{\partial \theta} = r \cos \theta$$

$$\begin{aligned} \therefore \frac{\partial(u, v)}{\partial(r, \theta)} &= \frac{\partial(u, v)}{\partial(x, y)} \cdot \frac{\partial(x, y)}{\partial(r, \theta)} \quad \because x^2 + y^2 = r^2 \\ &= \begin{vmatrix} \frac{\partial u}{\partial x} & \frac{\partial u}{\partial y} \\ \frac{\partial v}{\partial x} & \frac{\partial v}{\partial y} \end{vmatrix} \cdot \begin{vmatrix} \frac{\partial x}{\partial r} & \frac{\partial x}{\partial \theta} \\ \frac{\partial y}{\partial r} & \frac{\partial y}{\partial \theta} \end{vmatrix} \\ &= \begin{vmatrix} 2x & -2y \\ 2y & 2x \end{vmatrix} \cdot \begin{vmatrix} \cos \theta & -r \sin \theta \\ \sin \theta & r \cos \theta \end{vmatrix} \\ &= (4x^2 + 4y^2)(r \cos^2 \theta + r \sin^2 \theta) \\ &= 4(x^2 + y^2)(r) \\ &= 4r^2 \cdot r \\ &= 4r^3 \end{aligned}$$

H.W.P:-

(21)

- 1) If $u = x^2 - 2y$, $v = x + y + z$, $w = x - 2y + 3z$, find $\frac{\partial(u,v,w)}{\partial(x,y,z)}$. (or) (m.u.)
- 2) If $x = r \sin \theta \cos \phi$, $y = r \sin \theta \sin \phi$, $z = r \cos \theta$ find $\frac{\partial(x,y,z)}{\partial(r,\theta,\phi)}$. (or) (m.u.)
- 3) If $x = u(1-v)$, $y = uv$, p.t. $\frac{\partial(u,v)}{\partial(x,y)} \times \frac{\partial(x,y)}{\partial(u,v)} = 1$. (or) $JJ' = 1$

* FUNCTIONAL DEPENDENCE :- (OR) (Functional Relationship):-

Let $u = f(x,y)$ and $v = g(x,y)$ be two functions of differentiable in the independent variables x, y . Suppose these functions are connected by a relation $F(u,v) = 0$ where F is differentiable. Then we say that u and v are functionally dependent on one another if u_x, u_y, v_x, v_y are not all zero at a time.

i.e., Jacobian of u and v is zero $\Rightarrow u, v$ are functionally dependence. $J\left[\frac{u,v}{x,y}\right] = 0$.

otherwise, i.e., if $J\left[\frac{u,v}{x,y}\right] \neq 0 \Rightarrow u$ and v are called functionally independent.

* PROBLEMS:-

- 1) If $u = \frac{x+y}{1-xy}$, $v = \tan^{-1}x + \tan^{-1}y$, find $\frac{\partial(u,v)}{\partial(x,y)}$. Hence prove that u and v are functionally dependent. Find the relation functionally between them.

Sol:-

Given $u = \frac{x+y}{1-xy}$ and $v = \tan^{-1}x + \tan^{-1}y$

$$\frac{\partial u}{\partial x} = \frac{(1-xy) - (x+y)(-y)}{(1-xy)^2} = \frac{1-xy+xy+y^2}{(1-xy)^2} = \frac{1+y^2}{(1-xy)^2}$$

$$\frac{\partial u}{\partial y} = \frac{(1-xy) - (x+y)(-x)}{(1-xy)^2} = \frac{1-xy+x^2+xy}{(1-xy)^2} = \frac{1+x^2}{(1-xy)^2}$$

and $\frac{\partial v}{\partial x} = \frac{1}{1+x^2}$, $\frac{\partial v}{\partial y} = \frac{1}{1+y^2}$

$$\therefore \frac{\partial(u,v)}{\partial(x,y)} = \begin{vmatrix} \frac{1+y^2}{(1-xy)^2} & \frac{1+x^2}{(1-xy)^2} \\ \frac{1}{1+x^2} & \frac{1}{1+y^2} \end{vmatrix} = \frac{1}{(1-xy)^2} - \frac{1}{(1-xy)^2} = 0$$

$\therefore$ u and v are functionally dependence.

Now $v = \tan^{-1}x + \tan^{-1}y = \tan^{-1}\left[\frac{x+y}{1-xy}\right] = \tan^{-1}u$

$\therefore v = \tan^{-1}u$

which is functional relation b/w u and v .

* H.W.P:-

- 1) Show that the functions $u = xy + yz + zx$, $v = x^2 + y^2 + z^2$, $w = x + y + z$ are functionally related (ie F. depend). find relation b/w them.
- 2) Determine whether the following functions are functionally dependent (or) not. If F. Depend find relation b/w them
- (i) $u = e^{xyz}$, $v = e^{x^2y}$ (ii) $u = \frac{x}{y}$, $v = \frac{x+y}{x-y}$

* MAXIMUM AND MINIMUM OF FUNCTIONS OF TWO VARIABLES :-

* Defination :- Let $f(x, y)$ be a function of two variables x & y .
At $x=a, y=b$, $f(x, y)$ ~~has~~ is said to have maximum (or) minimum, if $f(a, b) > f(a+h, b+k)$ (or) $f(a, b) < f(a+h, b+k)$ respectively where h and k are small values.

* Extreme value :- $f(a, b)$ is said to be an extreme value of f , if it is a maximum (or) minimum value.

* Working rule to find Max. (or) Min. values of $f(x, y)$:-

Given $f(x, y)$ is a function of variables x & y .

(1) Find $\frac{\partial f}{\partial x}$ and $\frac{\partial f}{\partial y}$ and equate each to zero. solve these equations for x and y . Let $(a_1, b_1), (a_2, b_2), \dots$ be pairs of values.

(2) Find $\frac{\partial^2 f}{\partial x^2} = l, \frac{\partial^2 f}{\partial x \partial y} = m, \frac{\partial^2 f}{\partial y^2} = n$, for each pair of values obtained in step (1).

(3) (i) if $ln - m^2 > 0$ and $l < 0$ at (a_1, b_1) , then (a_1, b_1) is a point of maximum (i.e, f have maximum value at (a_1, b_1)) and $f(a_1, b_1)$ is maximum value.

(ii) if $ln - m^2 > 0$ and $l > 0$ at (a_1, b_1) , then (a_1, b_1) is a point of ~~max~~ minimum [i.e, f have minimum value at (a_1, b_1)] and $f(a_1, b_1)$ is minimum value.

(iii) if $ln - m^2 < 0$ at (a_1, b_1) , then $f(a_1, b_1)$ is not extreme value, i.e, there is neither a maximum nor a minimum at (a_1, b_1) . In this case (a_1, b_1) is "Saddle" point.

(iv) if $ln - m^2 = 0$ at, no conclusion can be drawn about max. (or) min. and needs further investigation.

Similarly, examine the other pair of values $(a_2, b_2), \dots$ one by one.

* STATIONARY VALUE :- $f(a,b)$ is said to be a stationary value of $f(x,y)$ if $f_x(a,b)=0$, $f_y(a,b)=0$ and (a,b) is stationary point.
Thus every extreme value a stationary value but the converse may not be true.

* PROBLEMS :-

1) Find the maximum and minimum values of x^3+y^3-3axy .

Sol:- let $f(x,y) = x^3+y^3-3axy$.

$$\text{We have } \frac{\partial f}{\partial x} = 3x^2 - 3ay = 0 \quad \text{--- (1)}$$

$$\frac{\partial f}{\partial y} = 3y^2 - 3ax = 0 \quad \text{--- (2)}$$

Solving eq (1) and (2),

$$\text{From (1), } x^2 = ay.$$

$$y = \frac{x^2}{a}$$

$$\text{, from (2), } \left(\frac{x^2}{a}\right)^2 = ax.$$

$$x^4 = a^3x.$$

$$x(x^3 - a^3) = 0.$$

$$x=0, x=a.$$

$$\text{So that } x=0 \Rightarrow y=0$$

$$x=a \Rightarrow y=a.$$

$\therefore$ we have the points $(0,0)$ and (a,a)

At the point $(0,0)$: Now $l = \frac{\partial^2 f}{\partial x^2} = 6x$

$$m = \frac{\partial^2 f}{\partial x \partial y} = -3a$$

$$n = \frac{\partial^2 f}{\partial y^2} = 6y$$

$$\text{and } \Delta n - m^2 = 36xy - 9a^2$$

At the point $(0,0)$:-

$$\text{at } (0,0), \Delta n - m^2 = -9a^2 < 0$$

$f(x,y)$ does not have any extreme value.

at the point (a, a) , $ln - m^2 = 36a^2 - 9a^2 = 27a^2 > 0$

and $l = 6a > 0$ if $a > 0$

$l = -6a < 0$, if $a < 0$

$\therefore f(x, y)$ have maximum at (a, a) if $a < 0$.

and minimum at (a, a) if $a > 0$.

$\therefore$ Maximum value $f(a, a) = a^3 + a^3 - 3a^3 = -a^3$

minimum values $f(a, a) = -a^3$

(2) Discuss the maxima and minima of $x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

Sol: let $f(x, y) = x^3 + 3xy^2 - 3x^2 - 3y^2 + 4$.

We have $\frac{\partial f}{\partial x} = 3x^2 + 3y^2 - 6x = 0$ — ①

$\frac{\partial f}{\partial y} = 6xy - 6y = 0$ — ②

From eq ②, $(x-1)y = 0$
 $x=1, y=0$

from ①, if $x=1 \Rightarrow 3 + 3y^2 - 6 = 0$
 $3y^2 = 3$
 $y = \pm 1$

if $y=0, \Rightarrow 3x^2 - 6x = 0$
 $3x(x-2) = 0$
 $x=0, x=2$

$\therefore$ we have the points are $(0, 0), (2, 0), (1, \pm 1)$

Now $l = \frac{\partial^2 f}{\partial x^2} = 6x - 6$

$m = \frac{\partial^2 f}{\partial x \partial y} = 6y$

$n = \frac{\partial^2 f}{\partial y^2} = 6x - 6$

and $ln - m^2 = (6x-6)^2 - 36y^2$

at the point $(0,0)$, $1n-m^2 = 36 > 0$
and $l = -6 < 0$

$\therefore f$ have maximum value at $(0,0)$
 $\therefore$ Maximum value $f(0,0) = 4.$

at the point $(2,0)$, $1n-m^2 = 36 > 0$
and $l = 6 > 0$

$\therefore f$ have minimum value at $(2,0)$
 $\therefore$ Minimum value $f(2,0) = 8 - 12 + 4$
 $= 0$

at $(1, \pm 1)$, $1n-m^2 < 0$
 $\therefore f(x,y)$ have no extreme values at $(1, \pm 1)$

(3) Find the stationary points of $u(x,y) = \sin x \sin y \sin(x+y)$
where $0 < x < \pi$, $0 < y < \pi$ and find the maximum u .

Sol: Given $u(x,y) = \sin x \sin y \sin(x+y)$, $0 < x, y < \pi$

$$\begin{aligned} \therefore \frac{\partial u}{\partial x} &= \cos x \sin y \sin(x+y) + \sin x \sin y \cos(x+y) \\ &= \sin y [\sin(x+y) \cos x + \cos(x+y) \sin x] \\ &= \sin y \sin(2x+y) \end{aligned}$$

we let $\frac{\partial u}{\partial x} = 0 \Rightarrow \sin y \sin(2x+y) = 0$
 $\sin y = 0$ (or) $\sin(2x+y) = 0$
 $y = 0, \pi \dots$ (or) $2x+y = \pi$ — (1)

$$\begin{aligned} \text{and } \frac{\partial u}{\partial y} &= \sin x \cos y \sin(x+y) + \sin x \sin y \cos(x+y) \\ &= \sin x \sin(2x+y) \end{aligned}$$

$$\frac{\partial u}{\partial y} = 0 \Rightarrow \sin x \sin(2x+y) = 0$$

$$\therefore x = 0 \text{ (or) } x+y = \pi \text{ — (2)}$$

Solving (1) and (2), we get $x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$
 $\therefore$ Stationary points $x = \frac{\pi}{3}$, $y = \frac{\pi}{3}$.

Now $l = \frac{\partial^2 U}{\partial x^2} = 2 \sin y \cos(2x+y)$

$m = \frac{\partial^2 U}{\partial x \partial y} = \sin x \cos(x+2y) + \cos x \sin(x+2y) = \sin(2x+y)$

$n = \frac{\partial^2 U}{\partial y^2} = 2 \sin x \cos(x+2y)$

and $\Delta n - m^2 = 4 \sin x \sin y \cos(x+2y) \cos(2x+y) - [\sin x \cos(x+2y) + \cos x \sin(x+2y)]^2$

at $(\frac{\pi}{3}, \frac{\pi}{3})$

$\Delta n - m^2 = 4 \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot (-1) \cdot (-1) - [\frac{\sqrt{3}}{2} \cdot (-1) + \frac{1}{2} \cdot (0)]^2$
 $= 3 - \frac{3}{4}$
 $= \frac{9}{4} > 0$

and $l = 2 \cdot \frac{\sqrt{3}}{2} \cdot (-1) = -\sqrt{3} < 0$

$\therefore u$ will be have maximum at $(\frac{\pi}{3}, \frac{\pi}{3})$
 $\therefore$ Maximum value of $u = u(\frac{\pi}{3}, \frac{\pi}{3})$

$= \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2} \cdot \frac{\sqrt{3}}{2}$
 $= \frac{3\sqrt{3}}{8}$

(5) Find the minimum value of $x^2+y^2+z^2$ given that $xyz=a^3$.
 at $(a,a), 3a^2$

* H.W.P:-

1) Discuss the maximum and minimum of $f(x,y) = x^2+y^2+6x+12$.

2) Test for maximum (or) minimum of $u(x,y) = x^2y^2-5x^2-8xy-5y^2$
 at $(0,0), (-1,1), (1,-1), (3,3), (-3,-3)$

3) Show that the function $f(x,y) = x^3+y^3-63(x+y)+12xy$ is 0
 maximum at $(-7,-7)$ and minimum at $(3,3)$.

4) Find the maximum and minimum values of
 $f(x,y) = xy + \frac{a^3}{x} + \frac{a^3}{y}$. at $(a,a), 3a^2$

5) Examine for minimum and maximum values of

$\sin x + \sin y + \sin(x+y)$. at $(\pm \frac{\pi}{3}, \pm \frac{\pi}{3}), \pm \frac{3\sqrt{3}}{2}$

(6) Find the shortest distance from origin to the surface $xyz^2=2$.

(4) A rectangular box open at the top is to have volume of 32 cubic ft. Find the dimensions of the box requiring least material for its construction. (5)

Sol: let x ft, y ft, z ft be the dimensions of the box and let S be the surface of the box.

Then we have $S = xy + 2yz + 2zx$ ($\because$ open at the top)

Given its volume $xyz = 32$ — (2)

$$z = \frac{32}{xy}$$

$$\therefore S = xy + 2y \cdot \frac{32}{xy} + 2 \cdot \frac{32}{xy} \cdot x$$

$$\therefore S(x, y) = xy + \frac{64}{x} + \frac{64}{y}$$

$$\text{Now } \frac{\partial S}{\partial x} = y + \left(-\frac{64}{x^2}\right) = 0 \Rightarrow$$

$$y - \frac{64}{x^2} = 0 \Rightarrow y = \frac{64}{x^2}$$

$$\text{and } \frac{\partial S}{\partial y} = x - \frac{64}{y^2} = 0 \Rightarrow$$

$$x - \frac{64}{y^2} = 0$$

$$x = \frac{64}{\left(\frac{64}{x^2}\right)^2}$$

$$x = \frac{x^4}{64}$$

$$x^4 - 64x = 0$$

$$x = 0, x = 4$$

$$\therefore x = 4 \quad (\because x = 0 \Rightarrow y = \infty)$$

$$\text{also } l = \frac{\partial^2 S}{\partial x^2} = \frac{+128}{x^3}, m = \frac{\partial^2 S}{\partial x \partial y} = 1, n = \frac{\partial^2 S}{\partial y^2} = \frac{128}{y^3}$$

$$\text{at } (4, 4), \quad ln - m^2 = \frac{128}{x^3} \cdot \frac{128}{y^3} - 1 = \frac{128 \times 128}{64 \times 64} - 1 = 4 - 1 = 3 > 0$$

$$\text{and } l = 2 > 0$$

$\therefore S$ will have minimum at $x = 4, y = 4$

$\therefore$ from (2), $z = 2$.

$\therefore$ The dimensions of the box for least material for its construction are 4, 4, 2.

Find the maximum distance of the point $(3, 4, 12)$ from the sphere $x^2 + y^2 + z^2 = 1$. * 5

Sol: let $Q(x, y, z)$ be any point on the surface $x^2 + y^2 + z^2 = 1$.

$\therefore$ distance b/w $P(3, 4, 12)$ and $Q(x, y, z)$ is

$$PQ^2 = (x-3)^2 + (y-4)^2 + (z-12)^2$$

$$\text{let } PQ^2 = x^2 - 6x + 9 + y^2 - 8y + 16 + z^2 - 24z + 144$$

$$\text{since given } x^2 + y^2 + z^2 = 1$$

$$x^2 = 1 - y^2 - z^2$$

$$\therefore \text{let } f(x, y) = PQ^2 = x^2 - 6x + y^2 - 8y + 1 - x^2 - y^2 - 24\sqrt{1 - x^2 - y^2} + 169$$

$$= 170 - 6x - 8y - 24\sqrt{1 - x^2 - y^2}$$

$$\text{Now } \frac{\partial f}{\partial x} = -6 - 24 \cdot \frac{1}{2\sqrt{1 - x^2 - y^2}} \cdot (-2x) = 0$$

$$-6 + \frac{24x}{\sqrt{1 - x^2 - y^2}} = 0$$

$$\frac{4x}{\sqrt{1 - x^2 - y^2}} = +1$$

$$16x^2 = 1 - x^2 - y^2$$

$$17x^2 + y^2 - 1 = 0 \quad \text{--- (1)}$$

$$\text{and } \frac{\partial f}{\partial y} = -8 - 24 \cdot \frac{1}{2\sqrt{1 - x^2 - y^2}} \cdot (-2y) = 0$$

$$\frac{24y}{\sqrt{1 - x^2 - y^2}} = +8$$

$$3y = +\sqrt{1 - x^2 - y^2}$$

$$9y^2 = 1 - x^2 - y^2$$

$$x^2 + 10y^2 - 1 = 0 \quad \text{--- (2)}$$

Solving (1) & (2)

$$170x^2 + 10y^2 - 10 = 0$$

$$x^2 + y^2 - 1 = 0$$

$$169x^2 - 9 = 0$$

$$x^2 = 9/169$$

$$x = \pm \frac{3}{13}$$

from (3),

$$\frac{9}{169} + 10y^2 - 1 = 0$$

$$10y^2 = \frac{160}{169}$$

$$y = \pm \frac{4}{13}$$

∴ The points are $(\pm \frac{3}{13}, \pm \frac{4}{13})$

$$\text{Now } l = \frac{\partial^2 f}{\partial x^2} = \frac{\sqrt{1-x^2-y^2} \cdot 24 - 24x \cdot \frac{1}{2\sqrt{1-x^2-y^2}} \cdot (-2x)}{(1-x^2-y^2)^{3/2}}$$

$$x^2 + y^2 = 1$$

$$(x, y) [x^2 - y^2] = 0$$

$$x = \pm y$$

$$\text{at } (\pm \frac{3}{13}, \pm \frac{4}{13})$$

$$\frac{(1-x^2-y^2) \cdot 24 + \frac{24x^2}{(1-x^2-y^2)^{3/2}}}{(1-x^2-y^2)^{3/2}}$$

$$\frac{24(1-y^2)}{(1-x^2-y^2)^{3/2}} > 0$$

$$\text{and } m = \frac{\partial^2 f}{\partial x \partial y} = \sqrt{1-x^2-y^2} \cdot \frac{-1}{2} \cdot \frac{-2y}{(1-x^2-y^2)^{3/2}} = \frac{24y^2}{(1-x^2-y^2)^{3/2}}$$

$$n = \frac{\partial^2 f}{\partial y^2} = \frac{24(1-x^2)}{(1-x^2-y^2)^{3/2}}$$

$$\text{and } l^2 - m^2 =$$

$$\frac{(24)^2 (1-x^2) (1-y^2)}{(1-x^2-y^2)^3} - \frac{(24)^2 y^4}{(1-x^2-y^2)^3}$$

$$= \frac{(24)^2 [1 + x^2 y^2 - y^2 - x^2 - y^4]}{(1-x^2-y^2)^3}$$

$$F = (x-3)^2 + (y-4)^2 + (z-12)^2 + \lambda(x^2 + y^2 + z^2 - 1)$$

$$\frac{\partial F}{\partial x} = 2(x-3) + 2\lambda x, \quad \frac{\partial F}{\partial y} = 2(y-4) + 2\lambda y$$

$$\frac{\partial F}{\partial z} = 2(z-12) + 2\lambda z$$

$$2x-6+2\lambda x=0, \quad 2y-8+2\lambda y=0$$

$$x-3+\lambda x=0, \quad y-4+\lambda y=0$$

$$x = \frac{3}{1+\lambda}, \quad y = \frac{4}{1+\lambda}, \quad z = \frac{12}{1+\lambda}$$

$$\frac{12y}{1+\lambda} + \frac{12}{1+\lambda} = 1$$

$$169 = (1+\lambda)^2$$

$$1+\lambda = \pm 13$$

$$\lambda = 12, \quad \lambda = -13$$

$$x = \frac{3}{13}, \quad y = \frac{4}{13}, \quad z = \frac{12}{13}$$

$$x = \frac{3}{13}, \quad y = \frac{4}{13}, \quad z = \frac{12}{13}$$

$$\therefore$$