UNIT- VI

VECTOR INTEGRAL CALCULUS

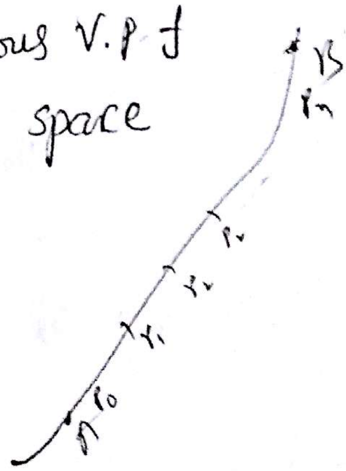
* **Smooth curve :-**

A curve C is smooth curve which is continuously differentiable.

* **Line Integral :-**

Any integral which is to be evaluated along a curve is called a line integral.

let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ be continuous V.P.f defined at every point of a curve C in space



Divide the curve C at the points $P_1, P_2 \ldots P_i, \ldots P_{n-1}$ and the end points being $A = P_0$ and $B = P_n$.

let $\bar{r}_0, \bar{r}_1 \ldots \bar{r}_i \ldots \bar{r}_n$ be the position vectors of the points $P_0, P_1 \ldots P_i, \ldots P_n$ respectively

Let ξ_i be a point on the arc $P_{i-1}P_i$, $i = 1 \to n$

Consider the sum $\sum_{i=1}^{n} \bar{f}(\xi_i) \cdot \delta\bar{r}_i$, where ——— (1)

$\delta\bar{r}_i = \bar{r}_i - \bar{r}_{i-1}$.

If the limit of the sum (1) as $n \to \infty$ (or) $|\delta\bar{r}_i| \to 0$ Exists, is called line integral of $\bar{f}$

along C and it is denoted by

$$\oint_C \bar{f}\cdot d\bar{r}$$

It is called the tangential line integral of $\bar{f}$ along C

* The other line integrals as $\oint_C \bar{f}\times d\bar{r}$ and $\oint_C \phi\, d\bar{r}$.

Cartesian form:-

let $\bar{f} = f_1\bar{i} + f_2\bar{j} + f_3\bar{k}$ and $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\therefore \bar{f}\cdot d\bar{r} = f_1\,dx + f_2\,dy + f_3\,dz$

Thus $\oint_C \bar{f}\cdot d\bar{r} = \int_C f_1\,dx + f_2\,dy + f_3\,dz$

If x, y, z are fns of t, then

$$\oint_C \bar{f}\cdot d\bar{r} = \oint_C \left(f_1 \frac{dx}{dt} + f_2 \frac{dy}{dt} + f_3 \frac{dz}{dt} \right) dt.$$

(or) $$\oint_C \bar{f}\cdot d\bar{r} = \oint_C \left(\bar{f}\cdot \frac{d\bar{r}}{dt} \right) dt$$

* Work done by force:-

If $\bar{f}$ represents the force acting on a particle moving along the arc AB. The workdone during the displacement $\delta\bar{r}$ is $\bar{f}\cdot d\bar{r}$

$\therefore$ The total workdone by $\bar{f}$ during displacement from A to B $= \int_A^B \bar{f}\cdot d\bar{r}$

If the force $\bar{f}$ is conservative, then

$$\bar{f} = \nabla\phi = \bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}$$

$$\therefore \int_A^B \bar{f}\cdot d\bar{r} = \int_A^B \left(\bar{i}\frac{\partial\phi}{\partial x} + \bar{j}\frac{\partial\phi}{\partial y} + \bar{k}\frac{\partial\phi}{\partial z}\right)\cdot(\bar{i}dx + \bar{j}dy + \bar{k}dz)$$

$$= \int_A^B d\phi$$

$$= \phi_B - \phi_A$$

Thus, the line integral is independent of the path, if $\bar{f}$ is conservative (or) irrotational.

i.e, $\bar{f} = \nabla\phi$.

Note:- If $\oint_C \bar{f}\cdot d\bar{r} = 0$, then $\bar{f}$ is irrotational (or) conservative, $\nabla\times\bar{f} = 0$.

i.e, no workdone is done and the energy is conserved.

PROBLEMS:-

1) If $\bar{F} = 3xy\bar{i} - y^2\bar{j}$, Evaluate $\oint_C \bar{F}\cdot d\bar{r}$, where c is the curve $y = 2x^2$ in the xy-plane from $(0,0)$ to $(1,2)$

Sol: Given $\bar{F} = 3xy\bar{i} - y^2\bar{j}$

curve $y = 2x^2 \Rightarrow dy = 4x\,dx$.

xy plane $\Rightarrow z = 0$

So that let position vector $\bar{r} = x\bar{i} + y\bar{j}$

$d\bar{r} = dx\bar{i} + dy\bar{j}$

$\therefore \bar{F}\cdot d\bar{r} = 3xy\,dx - y^2\,dy$

Since $y^2=2x \Rightarrow dy = 4x\,dx$, x limits $0 \to 1$.

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_C (3xy\,dx - y^2\,dy)$$

$$= \int_{x=0}^{1} (3x(x^2)\,dx - 4x^4 \cdot 4x\,dx)$$

$$= \int_0^1 (6x^3 - 16x^5)\,dx$$

$$= \left(\frac{6x^4}{4} - \frac{16x^6}{6}\right)_0^1$$

$$= \frac{3}{2} - \frac{8}{3}$$

$$= \frac{-7}{6}$$

(2) Find the work done by the force $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$ in moving particle from the point $(0,0,0)$ to $(1,1,1)$ along the curve $c: x=t, y=t^2; z=t^3$

Sol:- Given $\bar{F} = (3x^2 - 6yz)\bar{i} + (2y + 3xz)\bar{j} + (1 - 4xyz^2)\bar{k}$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{F}\cdot d\bar{r} = (3x^2 - 6yz)\,dx + (2y + 3xz)\,dy + (1 - 4xyz^2)\,dz$

The end points $(0,0,0)$ and $(1,1,1)$

curve: $x=t$, $y=t^2$, $z=t^3$

$dx = dt$. $dy = 2t\,dt$, $dz = 3t^2\,dt$

$\because t = 0 \to 1$

$\therefore$ Hence workdone $= \oint_C \bar{F}\cdot d\bar{r}$

$$= \int_{t=0}^{1} (3t^2 - 6t^5)\,dt + (2t^2 + 3t^4)\,2t\,dt + (1 - 4t^9)\,3t^2\,dt$$

$$= \int_0^1 (3t^2 - 6t^5 + 4t^3 + 6t^5 + 3t^2 - 12t^{11})\,dt$$

$$= \int_0^1 (6t^2 + 4t^3 - 12t^{11})\,dt$$

$$= \left(2t^3 + t^4 - t^{12}\right)_0^1$$

$$= 2$$

(3) If $\bar{F} = (4xy - 3x^2z^2)\bar{i} + 2x^2\bar{j} - 2x^3z\,\bar{k}$, show that $\oint_C \bar{F}\cdot d\bar{r}$, i.e, workdone is independent of the curve joining two points.

Sol: We know that, if the force is conservative i.e, $\bar{F} = \nabla\phi$, then the workdone is independent of the path and conversely.

So, we have to show that $\bar{F} = \nabla\phi$

i.e, $\text{curl}\,\bar{F} = \bar{0}$

$$\text{curl}\,\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 4xy - 3x^2z^2 & 2x^2 & -2x^3z \end{vmatrix}$$

$$= 0.$$

$\therefore$ The workdone is inde. of the path.

(4) Evaluate $\oint_C \bar{F}.d\bar{r}$, where $\bar{F} = 3x^2\bar{i} + (2xz - y)\bar{j} + z\bar{k}$ along the straight line C from (0,0,0) to (2,1,3).

Sol: The eq^n of line joining (0,0,0) and (2,1,3) is

$$\frac{x}{2} = \frac{y}{1} = \frac{z}{3} = t \text{ (say)}$$

Then along the line C: $x = 2t,\ y = t,\ z = 3t$, $t = 0 \rightarrow 1$

let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$

$\bar{r} = 2t\bar{i} + t\bar{j} + 3t\bar{k}$

$d\bar{r} = (2\bar{i} + \bar{j} + 3\bar{k})\,dt$

$\therefore \bar{F}.d\bar{r} = (3(4t^2)\bar{i} + (12t^2 - t)\bar{j} + 3t\bar{k}).$
$(2dt\,\bar{i} + dt\,\bar{j} + \bar{k}\,3dt)$

$= (24t^2 + 12t^2 - t + 9t)\,dt$

$= (36t^2 + 8t)\,dt$

$$\therefore \oint \bar{F}.d\bar{r} = \int_{t=0}^{1} (36t^2 + 8t)\,dt$$

$$= \left(\frac{36t^3}{3} + \frac{8t^2}{2}\right)_0^1$$

$= 12 + 4$

$= 16$

* Circulation :- Let C be a simple closed curve [(i,e) a curve which does not intersect it self any where]

The line Integral of $\bar{F}$ along C is called the circulation of $\bar{F}$ along C. It is denoted by $\oint_C \bar{F}.d\bar{r} = \oint_C (F_1dx + F_2dy + F_3dz)$

(4)

If $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$ evaluate $\oint_c \bar{F}.d\bar{r}$, where the curve c is the rectangle in the xy plane bounded by $y=0$, $y=b$, $x=0$, $x=a$.

Sol:- Given $\bar{F} = (x^2+y^2)\bar{i} - 2xy\bar{j}$

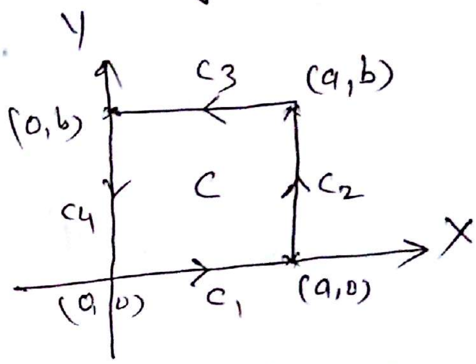
let $\bar{r} = x\bar{i} + y\bar{j} + z\bar{k}$, (where $z=0$. in xy-plane)

$$\oint_c \bar{F}.d\bar{r} = \oint_c (x^2+y^2)dx - 2xy\,dy \quad \text{——} (1)$$

Given curve c is rectangle bounded by

$y=0$, $y=b$, $x=0$, $x=a$

i.e, c is bounded by four paths c_1, c_2, c_3, c_4.



$$\therefore \oint_c \bar{F}.d\bar{r} = \oint_{c_1} \bar{F}.d\bar{r} + \oint_{c_2} \bar{F}.d\bar{r} + \oint_{c_3} \bar{F}.d\bar{r} + \oint_{c_4} \bar{F}.d\bar{r}$$

(i) Along c_1, $(0,0) \to (a,0)$

$y=0$, x varies $0 \to a$

$dy = 0$.

From (1),

$$\therefore \oint_{c_1} \bar{F}.d\bar{r} = \int_{x=0}^{a} x^2\,dx = \left(\frac{x^3}{3}\right)_0^a = \frac{a^3}{3}$$

(ii) Along c_2, $(a,0) \to (a,b)$

$x=a$, y varies $0 \to b$

F-(1), $dx = 0$

$$\oint_{c_2} \bar{F}.d\bar{r} = \int_{y=0}^{b} \cancel{(a^2+y^2)} - 2ay\,dy = -2a\frac{b^2}{2}$$

$$= -ab^2$$

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(iii) Along c_3, $(a,b) \to (0,b)$

$y = b$, $x = a \to 0$

$dy = 0$.

$$\oint_{c_3} \bar{F}\cdot d\bar{r} = \int_{x=a}^{0} (x^2+b^2)\,dx = \left(\frac{x^3}{3}+b^2x\right)_a^0$$

$$= -\frac{a^3}{3} - ab^2$$

(iv) Along c_4, $(0,b) \to (0,0)$

$x = 0$, $y = b \to 0$

$dx = 0$

$$\oint_{c_4} \bar{F}\cdot d\bar{r} = \int_{y=b}^{0} 0\, d = 0.$$

$\therefore$ Hence $\oint_C \bar{F}\cdot d\bar{r} = \frac{a^3}{3} - ab^2 - \frac{a^3}{3} - ab^2$

$$= -2ab^2$$

(6) If $\bar{F} = y\bar{i} + z\bar{j} + x\bar{k}$, find circulation of $\bar{F}$ round the curve C, where c is the circle $x^2+y^2=1$, $z=0$.

Sol:- Given equation to the curve

$x^2+y^2=1$, $z=0$.

$dz = 0$.

let $x = \cos\theta$, $y = \sin\theta$

$dx = -\sin\theta\, d\theta$, $dy = \cos\theta\, d\theta$

and θ varies $0 \to 2\pi$

The circulation of $\bar{F}$ along c is

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_C y\,dx + z\,dy + x\,dz$$

$$= \int_0^{2\pi} \sin\theta.(-\sin\theta)\,d\theta \quad (\sin^2\theta \text{ even})$$

$$= -4\int_0^{\pi/2} \frac{1-\cos 2\theta}{2}\,d\theta$$

$$= -\pi.$$

H.W.P

1) If $\bar{F} = xy\bar{i} - z\bar{j} + x^2\bar{k}$ and C is the curve $x = t^2, y = 2t,$ $z = t^3$ from $t=0$ to $t=1$, evaluate $\oint_C \bar{F}.d\bar{r}$.

2) If $\bar{F} = 2y\bar{i} - z\bar{j} + x\bar{k}$, evaluate $\oint \bar{F}.d\bar{r}$ over the C along the curve $x = \cos t, y = \sin t, z = 2\cos t$ from $t=0$ to $t = \pi/2$.

3) Evaluate $\int_{(0,1)}^{(1,2)} [(x^2 - y)dx + (y^2 + x)dy]$ along a straight line from $(0,1)$ to $(1,2)$

4) If C is simple closed curve in the xy-plane not enclosing the origin. show that $\oint_C \bar{F}.d\bar{r} = 0$, where $\bar{F} = \dfrac{y\bar{i} - x\bar{j}}{x^2+y^2}$

5) Evaluate the line integral $\oint_C (x^2 + yz)d\bar{r}$, where $C: x = t, y = t^2, z = 3t, 1 \le t \le 2$.

6) Evaluate $\oint_C (x^2+y^2)dx + (y+2x)dy$, where C is the boundary of the region in the first quadrant bounded by $y^2 = x$ and $x^2 = y$.

7) Evaluate $\oint_C f d\bar{R}$, where c is the curve $x = t, y = t^2,$ $z = t^3$, also $f = 2xy^2z + x^2y$ and $\bar{R} = x\bar{i} + y\bar{j} + z\bar{k}$.

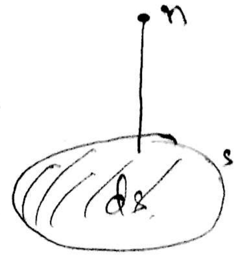
8) Find the circulation of $\bar{F} = (2x - y + 2z)\bar{i} + (x + y - z)\bar{j} + (3x - 2y - 5z)\bar{k}$ along the circle $x^2 + y^2 = 4, z = 0$ (8π)

Surface Integrals :-

Any integral which is to be evaluated over a surface is called a surface integral.

It is denoted $\int_S \bar{F}(\bar{r}).d\bar{A}$ (or) $\int_S \bar{F}.\bar{n}\,ds$ (or)

$\iint_S \bar{F}.d\bar{s}$, $d\bar{s}$ is orthogonal projection.



Here $d\bar{A} = \bar{n}\,ds$, $\bar{n}\,ds = d\bar{s}$

$\bar{n}$ is unit normal vector to surface.

$d\bar{A}$ is vector of area.

If the Vector field $\bar{F}$ represents the "flow of a ~~fluid~~ "fluid", then the surface integral of $\bar{F}$ will represent the amount of fluid flowing through the surface (per unit time).

The amount of the fluid flowing through the surface per unit time is also called the 'flux' of fluid through the surface.

For this reason, we often call the surface integral of a vector field a flux integral.

If water is flowing $\perp^r$ to the surface, a lot, water will flow through the surface and the flux will be large. On the other hand, if water is flowing parallel to the surface, water will not flow through the surface, and the flux will be zero. To calculate the total amount of water flowing through the surface, we want to add up the component of the vector $\bar{F}$ that is $\perp^r$ to the surface.

Let $\bar{n}$ be a unit normal vector to the surface.

The flux of fluid through the surface is determined by the component of $\bar{F}$, that is in the direction of $\bar{n}$,

i.e, $\bar{F} \cdot \bar{n}$

$\bar{F} \cdot \bar{n} = 0$, if $\bar{F}$ & $\bar{n}$ are $\perp^r$

> 0, if $\bar{F}$ & $\bar{n}$ are pointing same direction

< 0, if opposite direction.

a lot of ...

Cartesian form:- let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be vector function, where F_1, F_2, F_3 are continuous differentiable functions of x, y, z

Then
$$\int_S \bar{F}.\hat{n}\,ds = \iint_R (F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dy\,dx)$$

where $\hat{n}$ is unit normal to $\bar{n}$. S

Note:- (i) let R_1 be the projection of S on xy-plane

then
$$\int_S \bar{F}.\hat{n}\,ds = \iint_{R_1} \frac{\bar{F}.\hat{n}}{|\hat{n}.\bar{k}|}\,dx\,dy$$

(ii) let R_2 be the projection of S on yz-plane

then
$$\int_S \bar{F}.\hat{n}\,ds = \iint_{R_2} \frac{\bar{F}.\hat{n}}{|\hat{n}.\bar{i}|}\,dy\,dz$$

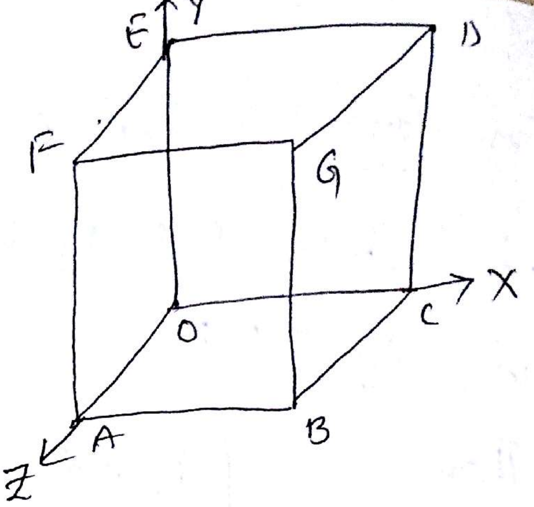
(iii) let R_3 be the projection of S on xz-plane

then
$$\int_S \bar{F}.\hat{n}\,ds = \iint_{R_3} \frac{\bar{F}.\hat{n}}{|\hat{n}.\bar{j}|}\,dx\,dz.$$

Note:- Some times given $\hat{n}$ is $\bar{N}$ is unit normal vector.

Thus surface integral S can be evaluated with the help of the double integral integrated over R_1 in xy-, yz, zx-planes.

Example:



Surface FGDE → donot on yz ($\because$ FE only line) and

xy (only line GD)

so, on xz

$\Rightarrow$ the region of projection in xz plane

will be the area of OABC.

Problems:-

1) Evaluate $\int_S \bar{F}\cdot\hat{n}\, ds$, where $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$

and S is the part of the plane $2x+3y+6z=12$

located in the first octant.

Sol: let $\phi = 2x+3y+6z-12$

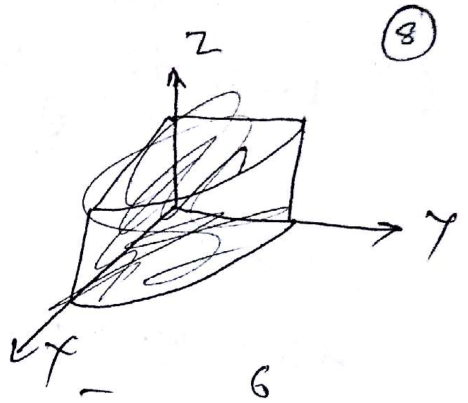
normal vector $\bar{n} = \nabla\phi = 2\bar{i} + 3\bar{j} + 6\bar{k}$

unit normal, $\hat{n} = \dfrac{\nabla\phi}{|\nabla\phi|} = \dfrac{2\bar{i}+3\bar{j}+6\bar{k}}{7}$

let R be the projection of S on XY-plane

then $\int_S \bar{F}\cdot\hat{n}\, ds = \iint_R \bar{F}\cdot\hat{n}\, \dfrac{dx\,dy}{|\hat{n}\cdot\bar{k}|}$

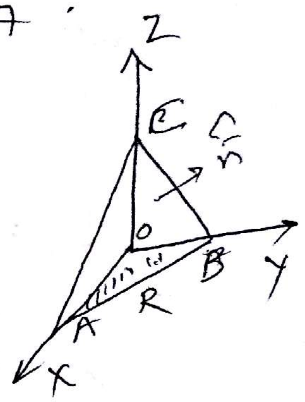
Give $\bar{F} = 18z\bar{i} - 12\bar{j} + 3y\bar{k}$



$$\therefore \bar{F}\cdot\hat{n} = \frac{1}{7}(36z + 36 + 18y)$$

$$= \frac{6}{7}(6z + 6 + 3y)$$

and $\hat{n}\cdot\bar{k} = \frac{1}{7}(2\bar{i} + 3\bar{j} + 6\bar{k})\cdot\bar{k} = \frac{6}{7}$.



Given surface $2x + 3y + 6z = 12$

$\therefore$ R of xy plane i.e $z = 0$

$$2x + 3y = 12$$

$$y = \frac{12 - 2x}{3}$$

($\because$ I octant)

$\therefore y = 0 \Rightarrow x = 6$

x varies $0 \to 6$, y varies $0 \to \frac{12-2x}{3}$

$\therefore$ Surface integral $\int_S \bar{F}\cdot\hat{n}\, ds = \int_{x=0}^{6}\left[\int_{y=0}^{\frac{12-2x}{3}} \frac{6}{7}(6z+6+3y)\cdot\frac{7}{6}\, dy\, dx\right]$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (6z + 6 + 3y)\, dy\, dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} (12 - 2x - 3y + 6 + 3y)\, dy\, dx$$

$$= \int_{x=0}^{6}\int_{y=0}^{\frac{12-2x}{3}} 2(3 - x)\, dy\, dx$$

$$= 2\int_{x=0}^{6} (y)_0^{\frac{12-2x}{3}} (3 - x)\, dx$$

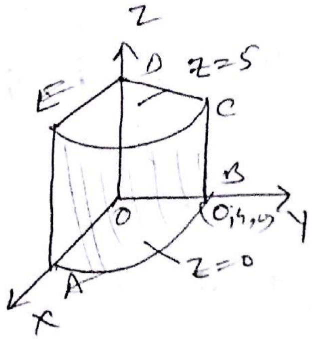
$$= 24$$

(2) Evaluate $\int_S \bar{F}\cdot\hat{N}\, d\bar{s}$, where $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$ and S is the surface cylinder $x^2+y^2=16$ included in the first octant between $z=0$ and $z=5$.

Sol: Given surface $\phi = x^2+y^2-16$

normal vector $= \nabla\phi$

$= 2x\bar{i} + 2y\bar{j}$



unit normal $\cdot\ \hat{N} = \dfrac{2x\bar{i}+2y\bar{j}}{\sqrt{4x^2+4y^2}}$

$= \dfrac{x\bar{i}+y\bar{j}}{4}$ $(\because x^2+y^2=16)$

let R be projection ~~vector~~ of S on yz-plane

then $\int_S \bar{F}\cdot\hat{N}\, ds = \iint_R \bar{F}\cdot\hat{N} \dfrac{dy\,dz}{|\hat{N}\cdot\bar{i}|}$

R is rectangle OBCD

Given $\bar{F} = z\bar{i} + x\bar{j} - 3y^2 z\bar{k}$

$\bar{F}\cdot\hat{N} = \frac{1}{4}(xy + xz)$ and $\hat{N}\cdot\bar{i} = \frac{x}{4}$

For the surface $x^2+y^2=16$ in yz plane, $x=0$

$\Rightarrow y=4$

in first octant, $y = 0 \to 4$

given $z = 0 \to 5$

Thus $\int_S \bar{F}\cdot\hat{N}\, ds = \int_{y=4}^{5}\int_{z=0}^{5} \frac{x}{4}(y+z)\cdot\frac{4}{x}\, dy\,dz$

$= \int_{y=1}^{5}\int_{z=0}^{5}(y+z)\, dz\, dy$

$= 90.$

) Evaluate $\int_S \bar{F}\cdot d\bar{s}$ where $\bar{F} = 4x\bar{i} - 2y^2\bar{j} - z^2\bar{k}$ and S is the surface bounding the region $x^2+y^2=4$, $z=0$, $z=3$

4) Evaluate $\int_S \bar{F}\cdot d\bar{s}$ over S where $\bar{F} = x\bar{i} + (z^2 - zx)\bar{j} - xy\bar{k}$ and S is the triangular surface with vertices $(2,0,0)$, $(0,2,0)$ & $(0,0,4)$

5) Evaluate $\iint_S (2x\bar{i} - 3y^2\bar{j} + z^2\bar{k})\cdot\hat{n}\,ds$ over the surface bounded by $x^2+y^2=1$ and $z=0$, $z=2$ by using Gauss divergence thm.

*

6) If $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ evaluate $\int_S \bar{F}\cdot\bar{N}\,ds$ over the surface $x^2+y^2+z^2=1$ in first octant.

3 Evaluate $\iint_S \bar{F}.\bar{n}\,ds$, where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ 7

and S is the part of the surface of the sphere $x^2+y^2+z^2=1$ which lies in the first octant.

Sol: let the surface $\phi = x^2+y^2+z^2-1$

$$\nabla\phi = 2[\bar{i}x + y\bar{j} + z\bar{k}]$$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

$$= x\bar{i} + y\bar{j} + z\bar{k}$$

let the R be the projection of the surface S on the xy-plane.

i.e, The region R is bounded by x-axis and y-axis and the circle $x^2+y^2=1$, $z=0$

$$\therefore \iint_S \bar{F}.\bar{n}\,ds = \iint_R \bar{F}.\bar{n}\,\frac{dx\,dy}{|\bar{n}.\bar{k}|}$$

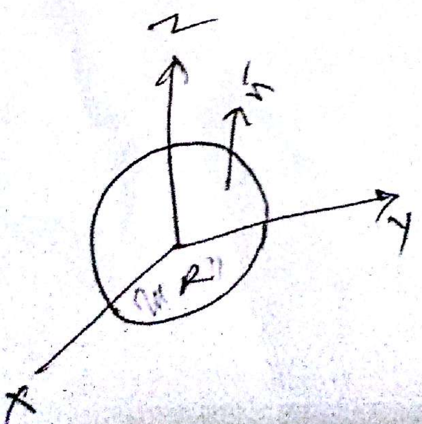
$\bar{F}.\bar{n} = 3xyz$, $\quad |\bar{n}.\bar{k}| = z$

$$\therefore \iint_S \bar{F}.\bar{n}\,ds = \iint_R 3xyz\,\frac{1}{z}\,dx\,dy$$

$$= \iint_R 3xy\,dx\,dy$$

let $x = r\cos\theta$, $y = r\sin\theta$,

$\theta = 0 \to \pi/2$, $r = 0 \to 1$

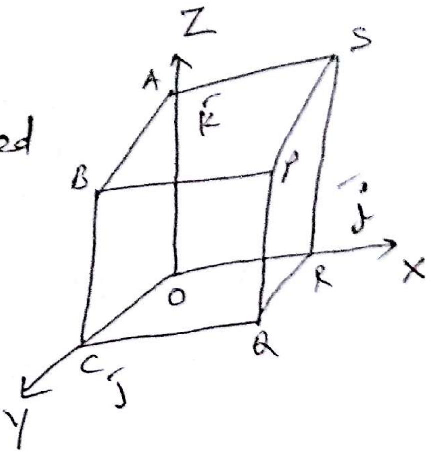


$$= \int_{\theta=0}^{\pi/2}\int_{r=0}^{1} 3\,r^2\sin^2\theta\; r\,dr\,d\theta$$

$$= \frac{3}{8}$$

If $\bar{F} = 4xz\bar{i} - y^2\bar{j} + yzk$, evaluate $\int \bar{F}.\bar{n}ds$, where S is the surface of the cube bounded by $x=0, x=a$, $y=0, y=a$, $z=0, z=a$

Sol: Consider the cube surrounded by the following faces (6) (six)



(i) for the face PQRS,

outward normal vector is $\bar{i}$

i.e, $\bar{n} = \bar{i}$, $x=a$, $ds = \frac{dydz}{|\bar{n}.\bar{i}|} = dydz$

[Projection is on yz-plane]

$$\therefore \int_{R_1} \bar{F}.\bar{n}ds = \int_{y=0}^{a}\int_{z=0}^{a} 4xz\, dy\, dz = \int_{y=0}^{a}\int_{z=0}^{a} 4az\, dydz$$

$$= 2a^4$$

(ii) for the face OABC.

outward normal vector is $-\bar{i}$

i.e, $\bar{n} = -\bar{i}$, $x=0$, $ds = \frac{dydz}{|\bar{n}.\bar{i}|} = dydz$

$$\int_{R_2} \bar{F}.\bar{n}\, ds = \int_{y=0}^{a}\int_{z=0}^{a} -4xz\, dydz = 0.$$

(iii) for the face BPQC,

outward normal vector is $\bar{j}$

i.e, $\bar{n} = \bar{j}$, $y=a$, $ds = \frac{dxdz}{|\bar{n}.\bar{j}|} = dxdz$

$$\int_{R_3} \bar{F}.\bar{n}ds = \int_{x=0}^{a}\int_{z=0}^{a} -y^2 . dxdz = -a^2.a.a = -a^4$$

(iv) For the face ORSA,

outward normal vector $\bar{n} = -\bar{j}$

$\therefore y = 0$, $ds = dx\,dz$

$$\int_{R_4} \bar{F}.\bar{n}\,ds = \int_{x=0}^{a} \int_{z=0}^{a} +y^2\,dx\,dz = 0.$$

(v) For the face ABPS,

Outward normal vector $\bar{n} = \bar{k}$

$\therefore z = a$, $ds = dx\,dy$

$$\int_{R_5} \bar{F}.\bar{n}\,ds = \int_{x=0}^{a} \int_{y=0}^{a} yz\,dx\,dy = \frac{a^4}{2}.$$

(vi) For the face OCQR,

outward normal vector $\bar{n} = -\bar{k}$

$\therefore z = 0$, $ds = dx\,dy$

$$\therefore \int_{R_6} \bar{F}.\bar{n}\,ds = \int_{x=0}^{a} \int_{y=0}^{a} -yz\,dx\,dy = 0.$$

$\therefore$ Hence $\int \bar{F}.\bar{n}\,ds = 2a^4 + 0 - a^4 + \frac{1}{2}a^4 + 0$

$$= \frac{3}{2}a^4$$

Volume Integral

let $\bar{F} = F_1\bar{i} + F_2\bar{j} + F_3\bar{k}$ be a vector point function where F_1, F_2, F_3 are fn's of x, y, z.

The volume integral of $\bar{F}$ in the region V is denoted by $\int_V \bar{F}\,dv$ (or) $\iiint_V (F_1\bar{i} + F_2\bar{j} + F_3\bar{k})\,dx\,dy\,dz$.

Problems:-

1) If $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2 k$, evaluate $\int_V \bar{F}\,dv$, where v is the region bounded by the surfaces $x=0, x=2, y=0, y=6, z=x^2, z=4$

Sol: Given $\bar{F} = 2xz\bar{i} - x\bar{j} + y^2 k$

The volume integral $\int_V \bar{F}\,dv = \iiint_V (2xz\bar{i} - x\bar{j} + y^2 k)\,dx\,dy\,dz$

$$= \int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} 2xz\,\bar{i}\,dx\,dy\,dz - \bar{j}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x^2}^{4} x\,dx\,dy\,dz + \bar{k}\int_{x=0}^{2}\int_{y=0}^{6}\int_{z=x}^{2} y^2\,dx\,dy\,dz$$

$$= 128\bar{i} - 24\bar{j} + 384\bar{k}$$

2) If $\bar{F} = (2x^2 - 3z)\bar{i} - 2xy\bar{j} - 4x\bar{k}$ evaluate (i) $\int_V \nabla\cdot\bar{F}\,dv$ and (ii) $\int_V \nabla\times\bar{F}\,dv$, where V is the closed region bounded by $x=0, y=0$; $z=0$, $2x + 2y + z = 4$

Sol: (i) $\nabla \cdot \bar{F} = 4x - 2x = 2x.$

Given $x=0, y=0, z=0, 2x+2y+z=4$

limits are $z=0, z = 4-2x-2y$.

" " $y=0, y = 2-x$

$x = 0, x=2$

$$\therefore \int_V \nabla \cdot F\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} 2x \, dx\, dy\, dz$$

$$= \frac{8}{3}$$

(ii) $\nabla \times \bar{F} = \bar{j} - 2y\bar{k}$

$$\therefore \int_V \nabla \times \bar{F}\, dv = \int_{x=0}^{2}\int_{y=0}^{2-x}\int_{z=0}^{4-2x-2y} (\bar{j} - 2y\bar{k})\, dz\, dy\, dx$$

$$= \frac{8}{3}(\bar{j} - \bar{k})$$

③ If $\phi = 45x^2y$, evaluate $\iiint_V \phi\, dv$, where v is the closed region bounded by the planes $4x+2y+z=8$; $x=0$, $y=0$, $z=0$. (128)

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Vector Integral Transforms (Theorems):-

(1) Green's theorem in plane:-

* Let S be a closed region in xy-plane enclosed by a simple closed curve C. and if M and N are continuously differentiable scalar functions of x and y in R. then

$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

where c is ~~trans~~ traversed in anti-clock (+ve) direction

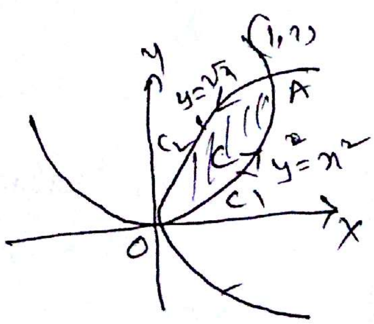
Problems:-

1) Verify Green's theorem in the plane for $\oint_c (3x^2-8y^2)\,dx + (4y-6xy)\,dy$ where c is the region bounded by $y=\sqrt{x}$ and $y=x^2$

Sol: Given $\oint (3x^2-8y^2)\,dx + \int (4y-6xy)\,dy$

$M = 3x^2-8y^2, \quad N = 4y-6xy$

$\frac{\partial M}{\partial y} = -16y \qquad \frac{\partial N}{\partial x} = -6y$



$y=\sqrt{x}$ & $y=x^2$

$\therefore \quad x = 0, 1$

$y = x^2 \to \sqrt{x}, \quad \forall x \in [0,1]$

By Green's Theorem

$$\oint_C M\,dx + N\,dy = \iint \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_{x=0}^{1}\int_{y=x^2}^{\sqrt{x}} (-6y + 16y)\,dy\,dx$$

$$= 10 \int_{x=0}^{1}\int_{x^2}^{\sqrt{x}} y\,dy\,dx$$

$$= 10 \int_{x=0}^{1} \left(\frac{y^2}{2}\right)_{x^2}^{\sqrt{x}} dx$$

$$= 10 \int_0^1 \left(\frac{x}{2} - \frac{x^4}{2}\right) dx$$

$$= 5 \left[\frac{x^2}{2} - \frac{x^5}{5}\right]$$

$$= \frac{3}{2}$$

Verification :-

The curve C is bounded two joint curves

$C_1 : \overline{OA}$ and $C_2 : \overline{AO}$

$$\therefore \oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy$$

$$= \int_{C_1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy + \int_{C_2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy \quad \text{--- (1)}$$

(i) along $C_1 : \overline{OA}$

$y = x^2$, x varies $0 \to 1$

$dy = 2x\,dx$

$$\therefore \int_{C_1} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dy = \int_{x=0}^{1} (3x^2 - 8x^4)\,dx + (4x^2 - 6x^3)\,2x\,dx$$

$$= \int_0^1 (3x^2 + 8x^3 - 20x^4)\,dx$$

$$= \left(x^3 + 2x^4 - 4x^5\right)_0^1$$

$$= -1$$

(ii) along $C_2 : \overline{AO}$, $y = \sqrt{x} \Rightarrow dy = \frac{1}{2\sqrt{x}}dx$, $x = 1 \to 0$

and $\int_{C_2} (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dx$

$$= \int_{x=1}^{0} (3x^2 - 8x)\,dx + (4\sqrt{x} - 6x\sqrt{x})\frac{1}{2\sqrt{x}}\,dx$$

$$= \int_1^0 (3x^2 - 8x + 2 - 3x)\,dx$$

$$= \left(x^3 - 4x^2 + 2x - \frac{3x^2}{2}\right)_1^0$$

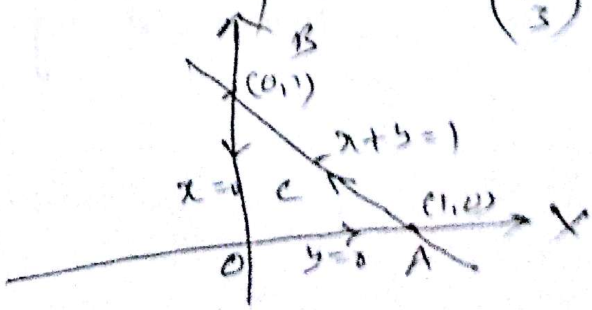
$$= -1 + 4 + 2 + \frac{3}{2}$$

$$= \frac{5}{2}$$

$\therefore \oint_C (3x^2 - 8y^2)\,dx + (4y - 6xy)\,dx = \frac{5}{2} + 1 = \frac{3}{2}$

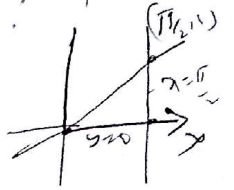
$\therefore$ Hence Theorem is verified.

(2) Verify Green's theorem for $\int_C (3x - 8y^2)\,dx + (4y - 6xy)\,dy$, where C is the boundary of the region bounded by $x = 0, y = 0, x + y = 1$. $\left(\frac{5}{3}\right)$



(3) Apply Green's theorem, evaluate $\int_C (y-\sin x)\,dx +$ (5) *
where C is the plane triangle enclosed by
the lines $y=0,\ x=\frac{\pi}{2},\ y=\frac{2x}{\pi}$

Sol: Given $\int_C (y-\sin x)\,dx + \cos x\,dy = \int_C M\,dx + N\,dy$



$M = y - \sin x$ ~~dx~~, $\quad N = \cos x$

$\frac{\partial M}{\partial y} = 1 \qquad \frac{\partial N}{\partial x} = -\sin x$

By Green's theorem

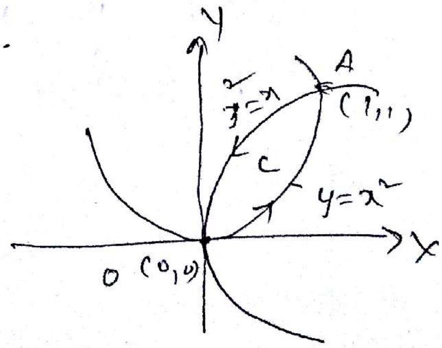
$$\oint_C M\,dx + N\,dy = \iint_S \left(\frac{\partial N}{\partial x} - \frac{\partial M}{\partial y}\right) dx\,dy$$

$$\int_C (y-\sin x)\,dx + \cos x\,dy = \iint_S (-\sin x - 1)\,dx\,dy$$

$$= \int_{x=0}^{x=\pi/2} \int_{y=0}^{2x/\pi} (-1-\sin x)\,dy\,dx$$

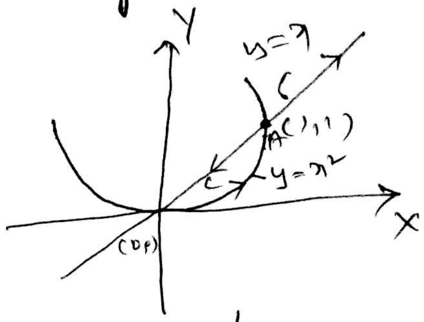
$$= -\left(\frac{\pi}{4} + \frac{2}{\pi}\right).$$

(4) Evaluate $\oint_C (x^2+y^2)\,dx + (y+2x)\,dy$, C is
* the boundary of the region in the first quadrant
bounded by $y^2=x$ and $x^2=y$.



(14)

Q. Verify green's theorem in the plane for $\int_C (xy+y^2)\,dx + x^2\,dy$, where C is the closed curve of the region bounded by $y=x$ & $y=x^2$



By G.T,

(6) Evaluate $\oint_C (3x+4y)\,dx + (2x-3y)\,dy$, where C is a circle $x^2+y^2=4$. (-8π)

$-2\iint_R dx\,dy$

$= -2\times\pi(2)^2$

$= -8\pi$

(7) By G.T, evaluate $\oint_C (x^2-2xy)\,dx + (x^2y+2)\,dy$ around the boundary C of the region defined by $y^2=8x$ & $x=2$. $(128/5)$

$\int_{-2}^{2}\int_{-\sqrt{4-x^2}}^{\sqrt{4-x^2}} dx\,dy$

$y=\sqrt{4-x^2}$

$\int_{-2}^{2} 3x+4\sqrt{4-x^2}+(2x-3\sqrt{4-x^2})\frac{1}{2\sqrt{4-x^2}}\cdot -2x$

$\int_{-2}^{2} 2\sqrt{4-x^2}\,dx$

$-4\left[\frac{x}{2}\sqrt{4-x^2}+\frac{4}{2}\sin^{-1}\frac{x}{2}\right]_{-2}^{2}$

$-4\left(0+2\frac{\pi}{2}+2\frac{\pi}{2}\right)$

-8π

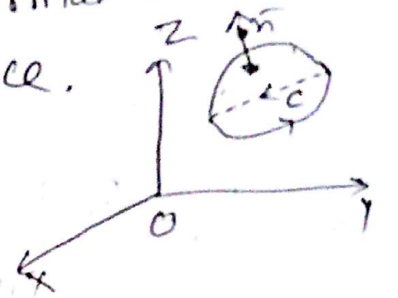
$3x+4\sqrt{4-x^2}-2x^2$

* STOKE'S THEOREM: (Transformation b/w lin & surface integral)

let S be an open surface bounded by a closed non-intersecting curve C. If $\bar{F}$ is any differentiable vpf. then

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S \operatorname{curl}\bar{F}\cdot\bar{n}\, ds$$

where c is traversed in the +ve direction and $\bar{n}$ is outward unit normal vector drawn at any point of the surface.



Problems:-

1) Verify stoke's theorem for $\bar{F} = (2x-y)\bar{i} - z^2y\bar{j} - zy^2\bar{k}$ where S is the upper half surface of the sphere $x^2+y^2+z^2=1$ and c is it's boundary.

(or)

Verify stoke's theorem for $\bar{F} = (2x-y)\bar{i} - z^2y\bar{j} - zy^2\bar{k}$ over the upper half surface of the sphere $x^2+y^2+z^2=1$ bounded by the projection of the xy-plane.

Sol: the boundary c of the surface S is the circle in xy-plane

i.e, $x^2+y^2=1$, $z=0$.

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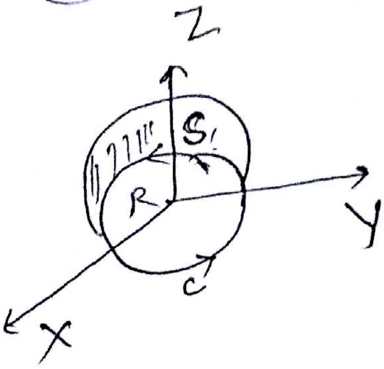


The parametric eqns are

$x = \cos\theta,\ y = \sin\theta \quad (\because r = 1)$

& $\theta : 0 \to 2\pi$

$dx = -\sin\theta\, d\theta,\ dy = \cos\theta\, d\theta$

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \int_C (2x-y)\,dx - z^2 y\,dy - z y^2 dz$$

$$= \int_{\theta=0}^{2\pi} (2\cos\theta - \sin\theta)(-\sin\theta)\,d\theta$$

$$= -\int_0^{2\pi} 2\cos\theta\sin\theta\, d\theta + \int_0^{2\pi} \frac{1-\cos 2\theta}{2}\, d\theta$$

$$= -\left(-\frac{\cos 2\theta}{2}\right)_0^{2\pi} + \frac{1}{2}\left[\theta - \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= \frac{1}{2}[1-1] + \frac{1}{4}[4\pi - 0]$$

$$= \pi.$$

Now
$$\nabla \times \bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 2x-y & -z^2 y & -z y^2 \end{vmatrix}$$

$$= \bar{k} \qquad (\because z = 0)$$

~~let~~ Let R be projection of S on xy-plane

$\therefore$ outward normal vector $\hat{n} =$

$\phi = x^2+y^2+z^2-1$

$\hat{n} = \dfrac{x\bar{i}+y\bar{j}+z\bar{k}}{\quad}$

$(\nabla\times\bar{F})\cdot\bar{n} = z$

$$ds = \frac{dx\,dy}{|\hat{n}\cdot\bar{k}|} = \frac{dx\,dy}{z}$$

(7)

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \iint_R 1\cdot dx\,dy = \pi \quad (\because \pi r^2)$$

Total surface area on circle $x^2+y^2=1,\ z=0$

$$\therefore = 4\int_{x=0}^{1}\int_{y=0}^{y=\sqrt{1-x^2}} dy\,dx$$

$$= 4\int_{x=0}^{1}\sqrt{1-x^2}\,dx$$

$$= 4\left[\frac{x}{2}\sqrt{1-x^2}+\frac{a}{2}\sin^{-1}x\right]_0^1$$

$$= 4\cdot\frac{\pi}{4}$$

$$= \pi$$

Stoke's theorem verified.

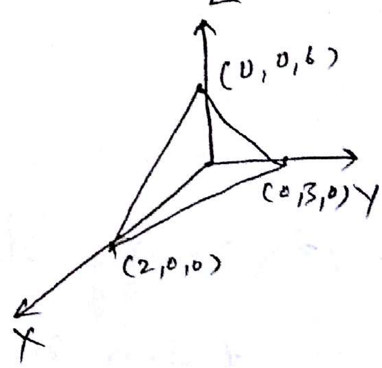
Using stokes theorem evaluate $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$, where c is the boundary of the traingle with vertices $(2,0,0)$, $(0,3,0)$ and $(0,0,6)$

Sol: let $\int_C (x+y)dx + (2x-z)dy + (y+z)dz$

$$= \int_C \left(\bar{i}(x+y) + (2x-z)\bar{j} + (y+z)\bar{k}\right) \cdot d\bar{r}$$

let $\bar{F} = \bar{i}(x+y) + (2x-z)\bar{j} + (y+z)\bar{k}$

stokes theorem $\int_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot\bar{n}\, ds$



Surface $\frac{x}{2} + \frac{y}{3} + \frac{z}{6} = 1$

$3x + 2y + z = 6$

$\therefore$ let $\cancel{\phi = 3x+2y+z = 6}$

$\cancel{\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{3\bar{i}+2\bar{j}+\bar{k}}{\sqrt{14}}}$

$$\nabla\times\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ x+y & 2x-z & y+z \end{vmatrix}$$

$$\bar{n} = \frac{3\bar{i}+2\bar{j}+\bar{k}}{\sqrt{9+4+1}} = \frac{1}{\sqrt{14}}(3\bar{i}+2\bar{j}+\bar{k})$$

$$= \bar{i}[2] - \bar{j}[0-0] + \bar{k}[2-1]$$

$$= 2\bar{i} + \bar{k}$$

$$\nabla\times\bar{F}\cdot\bar{n} = \frac{6}{\sqrt{14}} + \frac{1}{\sqrt{14}} = \frac{7}{\sqrt{14}}$$

let R be projection to surface on xy-plane $z=0$

$\bar{n} = \bar{k}$ $\quad \bar{n}\cdot\bar{k} = \frac{1}{\sqrt{14}}$

$ds = dx\,dy$

$$\int_S \bar{F}\cdot\bar{n}\,ds = \iint_R 7\,dx\,dy$$

$$= 7\int_{x=0}^{2}\int_{y=0}^{\frac{6-3x}{2}} dx\,dy$$

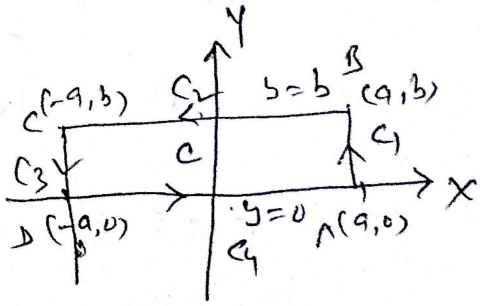
$$= 7\int_0^2 \frac{6-3x}{2}\,dx$$

$$= \frac{7}{2}\left(6x - \frac{3x^2}{2}\right)_0^2$$

$$= \frac{7}{2}\,[12-6]$$

$$= 21$$

③ Verify stokes theorem for $\bar{F} = (x^2+y^2)\,\bar{i} - 2xy\,\bar{j}$ taken round the the rectangle bounded by the lines $x = \pm a,\ y=0,\ y=b$.



Sol: let ABCD rectangle.

$C_1 : \overline{AB} : x = a$

$C_2 : \overline{BC} : y = b$

$C_3 : \overline{CD} : x = -a$, $C_4\ \overline{DA} : y = 0$

$\therefore$ stokes theorem :

$$\oint_C \bar{F}\cdot d\bar{r} = \int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

LHS:

$$\oint_C \bar{F}\cdot d\bar{r} = \oint_{C_1} \bar{F}\cdot d\bar{r} + \oint_{C_2} \bar{F}\cdot d\bar{r} + \oint_{C_3} \bar{F}\cdot d\bar{r} + \oint_{C_4} \bar{F}\cdot d\bar{r}$$

$$= -4ab^2$$

RHS:

$$\int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds$$

$$\nabla\times\bar{F} = -4y\bar{k}$$

let the outward unit normal vector to xy-plane

$$\bar{n} = \bar{k}$$

$$\therefore ds = \frac{dy\,dx}{|\bar{n}\cdot\bar{k}|} = dx\,dy$$

$$(\nabla\times\bar{F})\cdot\bar{n} = -4y$$

$$\therefore \int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \iint_R -4y\,dx\,dy$$

$$= \int_{x=-a}^{a}\int_{y=0}^{b} -4y\,dy\,dx$$

$$= -4\left(\frac{y^2}{2}\right)_0^b \cdot (x)_{-a}^{a}$$

$$= -4ab^2$$

$\therefore$ Verified.

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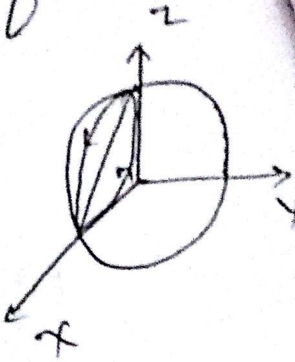
(4) using stokes theorem evaluate $\int_C y\,dx+z\,dy+x\,dz$ where c is the curve of intersection of $x^2+y^2+z^2=a^2$ and $x+z=a$



Sol: Stokes theorem

$$\oint_c \bar F\cdot d\bar r = \int_S (\nabla\times \bar F)\cdot \bar n\, ds$$

let $\int_c \bar F\cdot d\bar r = \int_c y\,dx+z\,dy+x\,dz$

$\therefore \bar F = y\bar i + z\bar j + x\bar k$

$\nabla\times\bar F = -(\hat i+\hat j+\hat k)$

let the projection be taken in xy-plane.

$\bar n = \frac{\nabla\phi}{|\nabla\phi|}$, $\phi = x^2+y^2+z^2-a^2$ ∴ $\cancel{\bar n = k}$

$= \frac{x}{a}\bar i+\frac{y}{a}\bar j+\frac{z}{a}\bar k$ $\int_S (\nabla\times\bar F)\cdot\bar n\, ds$

$= \int_S -(\bar i+\bar j+\bar k)\cdot\bar k\, ds$

$|\bar n\cdot\bar k| = \frac{z}{a}$

$ds = \frac{dx\,dy}{z/a}$

$= -\int_S ds = -4\pi a^2$

[∵ surface are of sphere is $4\pi a^2$]

$\nabla f\cdot\bar n$ $= -z/a$

$\therefore \int_c \bar F\cdot d\bar r = -4\pi a^2$

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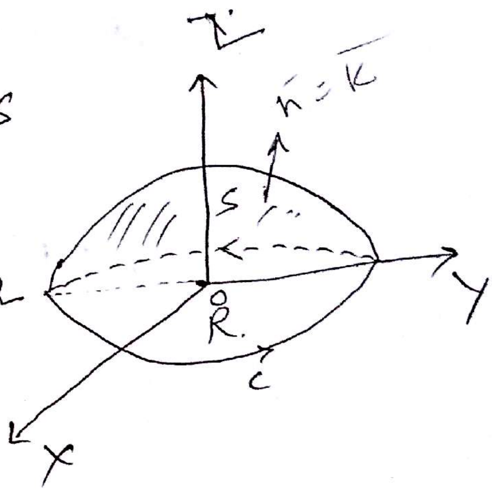
$z\,dz$ $x \times x\,dz$

Verify Stokes theorem for the vector field $\bar{f} = (3x-y)\hat{i} - 2yz^2\hat{j} - 2y^2z\hat{k}$, where S is the surface of the sphere $x^2+y^2+z^2=16$, $z>0$.

Sol: Consider the projection of S on the xy plane.



The projection is the circular region $x^2+y^2 \le 16$, $z=0$

and the bounding curve C is the circle $x^2+y^2=16$, $z=0$.

$$\therefore \oint_C \bar{F}\cdot d\bar{r} = \oint_C (3x-y)dx - 2yz^2dy - 2y^2z\,dz$$

$$= \oint_C (3x-y)\,dx \qquad (\because z=0)$$

let $x = 4\cos\theta$, $y = 4\sin\theta$, $0 \le \theta \le 2\pi$

$$= \int_0^{2\pi} 4(3\cos\theta - \sin\theta)\cdot -4\sin\theta\,d\theta$$

$$= -16\int_0^{2\pi} \left(\frac{3}{2}\sin 2\theta - \frac{1-\cos 2\theta}{2}\right)d\theta$$

$$= -8\left[\frac{-3\cos 2\theta}{2} - \theta + \frac{\sin 2\theta}{2}\right]_0^{2\pi}$$

$$= -8\left[-\frac{3}{2} + \frac{3}{2} - 2\pi\right]$$

$$= 16\pi$$

Given surface $\phi = x^2+y^2+z^2-16$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = \frac{1}{4}(x\hat{i}+y\hat{j}+z\hat{k})$$

R is projection on xy-plane.

$$ds = \frac{dx\,dy}{|\bar{n}\cdot\hat{k}|} = \frac{dx\,dy}{(z/4)}$$

$$\nabla\times\bar{F} = \begin{vmatrix} \hat{i} & \hat{j} & \hat{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ 3x-y & -2yz^2 & -2y^2z \end{vmatrix}$$

$$= \bar{k}$$

$$\therefore (\nabla\times\bar{F})\cdot\bar{n} = \frac{z}{4}$$

$$\therefore \int_S (\nabla\times\bar{F})\cdot\bar{n}\,ds = \iint_R \frac{z}{4}\,\frac{dx\,dy}{(z/4)}$$

$$= \iint_R dx\,dy$$

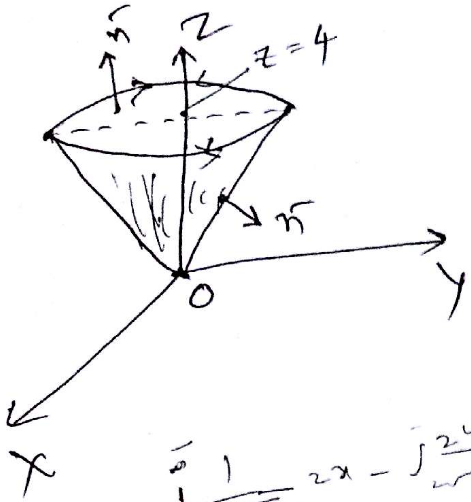
$$= \pi(4)^2$$

$$= 16\pi \qquad (\because \pi r^2)$$

which is the area of the circular region in the xy-plane.

$\therefore$ Stoke's theorem verified.

(6) Evaluate $\oint_C 2y^3 dx + x^3 dy + z\,dz$. where C is the trace of the cone $z = \sqrt{x^2+y^2}$ intersected by the plane $z = 4$ & S is the surface of the cone below $z = 4$



Sol:

$\bar{F} = 2y^2\,\bar{i} + x^3\,\bar{j} + z\,\bar{k}$

$\nabla \times \bar{F} = \bar{k}\,(3x^2 - 6y^2).$

unit normal vector

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|}$$

where $\phi = \sqrt{x^2+y^2} - z$

$$\nabla\phi = \frac{x\bar{i} + y\bar{j} - z\bar{k}}{z}$$

$-\bar{i}\,\frac{1}{2\sqrt{x^2+y^2}}\,2x - \bar{j}\,\frac{2y}{\ldots} + 1$

$= -ix$

$$\bar{n} = \frac{x\bar{i} + y\bar{j} - z\bar{k}}{\sqrt{2}\,z}$$

let R be projection of S on xy-plane.

$x^2+y^2 = z^2$, $z = 0$, $z = 4$

$$ds = \frac{dx\,dy}{|\bar{n}\cdot\bar{k}|} = -\sqrt{2}\,dx\,dy$$

$$(\nabla\times\bar{F})\cdot\bar{n} = -\frac{1}{\sqrt{2}}(3x^2 - 6y^2).$$

$$= \int_S \nabla\times\bar{F}\cdot\bar{n}\,ds = \iint_R (3x^2 - 6y^2)\,dx\,dy$$

let $x = r\cos\theta$, $y = r\sin\theta$

$r = 0 \to 4$, $\theta = 2\pi \to 0$

$= 192\pi.$

* Evaluate By Stokes theorem $\int_C \bar{F}\cdot d\bar{r}$, where $\bar{F} = yz\bar{i} + zx\bar{j} + xy\bar{k}$ and C is the curve $x^2+y^2=1$, $z=y^2$

Sol:

$$\text{curl } \bar{F} = \bar{0}$$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = 0.$$

* Evaluate $\int_C \sin z\, dx - \cos x\, dy + \sin y\, dz$

c is boundary of the rectangle

$0 \le x \le \pi$, $0 \le y \le 1$, $z = 3$

Sol:

$$\text{curl}.\bar{F} = \begin{vmatrix} \bar{i} & \bar{j} & \bar{k} \\ \frac{\partial}{\partial x} & \frac{\partial}{\partial y} & \frac{\partial}{\partial z} \\ \sin z & -\cos x & \sin y \end{vmatrix}$$

$$= \cos y\,\bar{i} + \cos z\,\bar{j} + \sin x\,\bar{k}$$

the rectangle lie in xy plane

$\therefore \bar{n} = \bar{k}$

$ds = dx\,dy$

$$\therefore \int_C \bar{F}\cdot d\bar{r} = \iint_R \nabla\times\bar{F}\cdot\bar{n}\,ds$$

$$= \int_{y=0}^{1}\int_{x=0}^{\pi} \sin x\, dx\, dy$$

$$= 2$$

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GAUSS'S Divergence Theorem:-

Let S be a closed surface enclosed a volume V. If $\bar{F}$ is a continuously differentiable v.p.f. then

$$\int_V \operatorname{div} \bar{F}\, dV = \int_S \bar{F}\cdot\bar{n}\, ds$$

where $\bar{n}$ is the outward drawn unit normal vector at any point of S.

Problems:-

1) Verify Gauss's divergence theorem to evaluate $\int_S ((x^3-yz)\bar{i} - 2x^2y\,\bar{j} + z\bar{k})\cdot\bar{n}\,ds$ over the surface of a cube bounded by the coordinates planes $x=y=z=a$

Sol: Given $\bar{F} = (x^3-yz)\bar{i} - 2x^2y\,\bar{j} + z\bar{k}$

G.D.T $\int_V \operatorname{div}\bar{F}\,dv = \int_S \bar{F}\cdot\bar{n}\,ds.$

$$\nabla\cdot\bar{F} = (3x^2 - 2x^2 + 1)$$

$$\therefore \int_V \operatorname{div}\bar{F}\,dv = \int_{x=0}^{a}\int_{y=0}^{a}\int_{z=0}^{a} (x^2+1)\,dx\,dy\,dz$$

$$= \left(\frac{x^3}{3}+x\right)_0^a (a)(a)$$

$$= \frac{a^5}{3} + a^3$$

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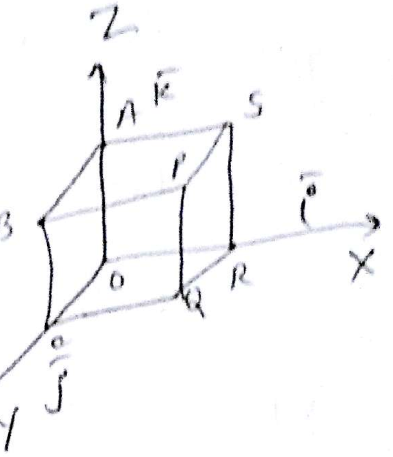
Verification



(i) for face PQRS,

$\bar{n} = \bar{i}, \quad ds = \frac{dy\,dz}{|\bar{n}.\bar{i}|} = dy\,dz$

$x = a$

$$\int_{S_1} \bar{F}.\bar{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} (x^3 - yz)\,dy\,dz$$

$$= a^5 - \frac{1}{4}a^4$$

(ii) for face OABC, $\quad \bar{n} = -\bar{i}, \quad ds = dy\,dz, \ x=0$

$$\int_{S_2} \bar{F}.\bar{n}\,ds = \int_{y=0}^{a}\int_{z=0}^{a} -(x^3 - yz)\,dy\,dz$$

$$= \frac{1}{4}a^4$$

(iii) for face ABPS, $\quad \bar{n} = \bar{k}, \quad ds = dx\,dy, \ z=a$

$$\int_{S_3} \bar{F}.\bar{n}\,ds = \int_{x=0}^{a}\int_{y=0}^{a} z\,dx\,dy = a^3$$

(iv) for face OCQR, $\quad \bar{n} = -\bar{k}, \quad ds = dx\,dy, \ z=0$

$$\int_{S_4} \bar{F}.\bar{n}\,ds = 0.$$

(v) for face BCQP, $\quad \bar{n} = \bar{j}, \ ds = dx\,dz, \ y=a$

$$\int_{S_5} \bar{F}.\bar{n}\,ds = \int_{x=0}^{a}\int_{z=0}^{a} -2x^2y\,dx\,dz = \frac{-2}{3}a^5$$

(vi) for face ORSA, $\quad n = -\bar{j}, \ ds = dx\,dz, \ y=0$

$$\int_{S_6} \bar{F}.\bar{n}\,ds = 0.$$

$$\therefore \int_{S} \bar{F}.\bar{n}\,ds = \frac{1}{3}a^5 + a^3.$$

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Q. By transforming into triple integral, evaluate $\iint_S (x^3\,dy\,dz + x^2y\,dz\,dx + x^2z\,dx\,dy)$, where S is the closed surface consisting of the cylinder $x^2+y^2=a^2$ and the circular discs $z=0$ and $z=b$.

Sol: Using Gauss's D.T.

Given $\iint_S F_1\,dy\,dz + F_2\,dx\,dz + F_3\,dx\,dy$

$\therefore F_1 = x^3,\ F_2 = x^2y,\ F_3 = x^2z$

let $\bar{F} = x^3\bar{i} + x^2y\bar{j} + x^2z\bar{k}$

$\nabla\cdot\bar{F} = 3x^2 + x^2 + x^2 = 5x^2$

$\therefore \int_V \nabla\cdot\bar{F}\,dv = \iiint_V 5x^2\,dx\,dy\,dz$

Given region cylinder $x^2+y^2=a^2$, i.e., in xy plane & $z=0, z=b$.

$x=0,\ y=0$ and $y=\sqrt{a^2-x^2}$ (or) $y=\pm\sqrt{a^2-x^2}$

$y=0 \Rightarrow x=a$ $\qquad x=\pm a$

soltn- $y=0 \rightarrow \sqrt{a^2-x^2}$

$z=0 \rightarrow b$

$x=0 \rightarrow a$

in the first quadrant

$$\therefore \quad = 4 \int_{z=0}^{b} \int_{x=0}^{a} \int_{y=0}^{\sqrt{a^2-x^2}} 5x^2 \, dz\, dx\, dy$$

$$= 20 \int_{z=0}^{b} \int_{x=0}^{a} x^2 \sqrt{a^2-x^2} \, dx\, dz$$

$$= 20b \int_{x=0}^{a} x^2 \sqrt{a^2-x^2} \, dx.$$

put $x = a\sin\theta$, $dx = a\cos\theta \, d\theta$

$$= 20\, b \int_0^{\pi/2} a^2\sin^2\theta \; \cos^2\theta \; a^2 d\theta$$

$$= 20a^4 b \int_0^{\pi/2} \sin^2\theta \,(1-\sin^2\theta)\, d\theta$$

$$= 20a^4 b \int_0^{\pi/2} (\sin^2\theta - \sin^4\theta)\, d\theta$$

$$= 20a^4 b \left[\frac{1}{2}\cdot\frac{\pi}{2} - \frac{3}{4}\cdot\frac{1}{2}\cdot\frac{\pi}{2} \right]$$

$$= \frac{5\pi a^4 b}{4}$$

$$\left(\int_0^{\pi/2} \sin^n\theta = \frac{n-1}{n}\cdot\frac{n-3}{n-2} \cdots \frac{\pi}{2} \right)$$

n is even

By G.D.T, evaluate $\int_S (ax^2+by^2+cz^2)\,ds$ over the surface $x^2+y^2+z^2=1$

Sol:

$$\int_S \bar{F}\cdot\bar{n}\,ds = \int_V \operatorname{div}\bar{F}\,dv$$

Given, let $\bar{F}\cdot\bar{n} = ax^2+by^2+cz^2$

& surface $\phi = x^2+y^2+z^2=1$

$$\bar{n} = \frac{\nabla\phi}{|\nabla\phi|} = x\bar{i}+y\bar{j}+z\bar{k}$$

$$\therefore \bar{F}\cdot\bar{n} = \bar{F}\cdot(x\bar{i}+y\bar{j}+z\bar{k}) = ax^2+by^2+cz^2$$

$$\therefore \bar{F} = ax\bar{i}+by\bar{j}+cz\bar{k}$$

$$\therefore \operatorname{div}\bar{F} = a+b+c$$

$$\therefore \int_S (ax^2+by^2+cz^2)\,ds = \int_V (a+b+c)\,dv$$

$$= (a+b+c)\frac{4\pi}{3}$$

where $\int_V dv$ is volume of surface $x^2+y^2+z^2=a^2$ is $\frac{4\pi}{3}r^3$

(4) Verify Gauss theorem for $\bar{F} = x^3\bar{i}+y^3\bar{j}+z^3\bar{k}$ and ~~xx~~ where S is surface of the sphere $x^2+y^2+z^2=a^2$

Sph. P. C

$x = r\cos\theta\sin\phi$

$y = r\sin\theta\sin\phi$

$z = r\cos\phi$

Sol:

$$\nabla\cdot F = 3x^2+3y^2+3z^2$$

$$\int_V \nabla\cdot F\,dv = \iiint_V 3(r^2)\,dx\,dy\,dz$$

$$= \int_{\theta=0}^{2\pi}\int_{\phi=0}^{\pi}\int_{r=0}^{a} 3r^2\cdot r^2\sin\phi\,dr\,d\phi\,d\theta$$

$$= \frac{12\pi a^5}{5}$$

8

(5) Evaluate $\iint_S (2x\hat{i} - 3y^2\hat{j} + z^2\hat{k}) \cdot \bar{n}\, ds$ over the surface bounded by $x^2+y^2=1$, $z=0$, $z=2$ by G. D. T

Sol

$\bar{F} = 2x\hat{i} - 3y^2\hat{j} + z^2\hat{k}$

$\nabla \cdot F = \cancel{2\hat{i} - 6y\hat{j} + 2z\hat{k}}$

$= 2 - 6y + 2z$

$\iint_S F \cdot \bar{n}\, ds = \int_V \mathrm{Div}\,\bar{F}\, dv$

$= \int_{z=0}^{2} \int_{x=-1}^{x=1} \int_{y=-\sqrt{1-x^2}}^{\sqrt{1+x^2}} (2 - 6y + 2z)\, dx\, dy\, dz.$